APPLIED MATHEMATICS III

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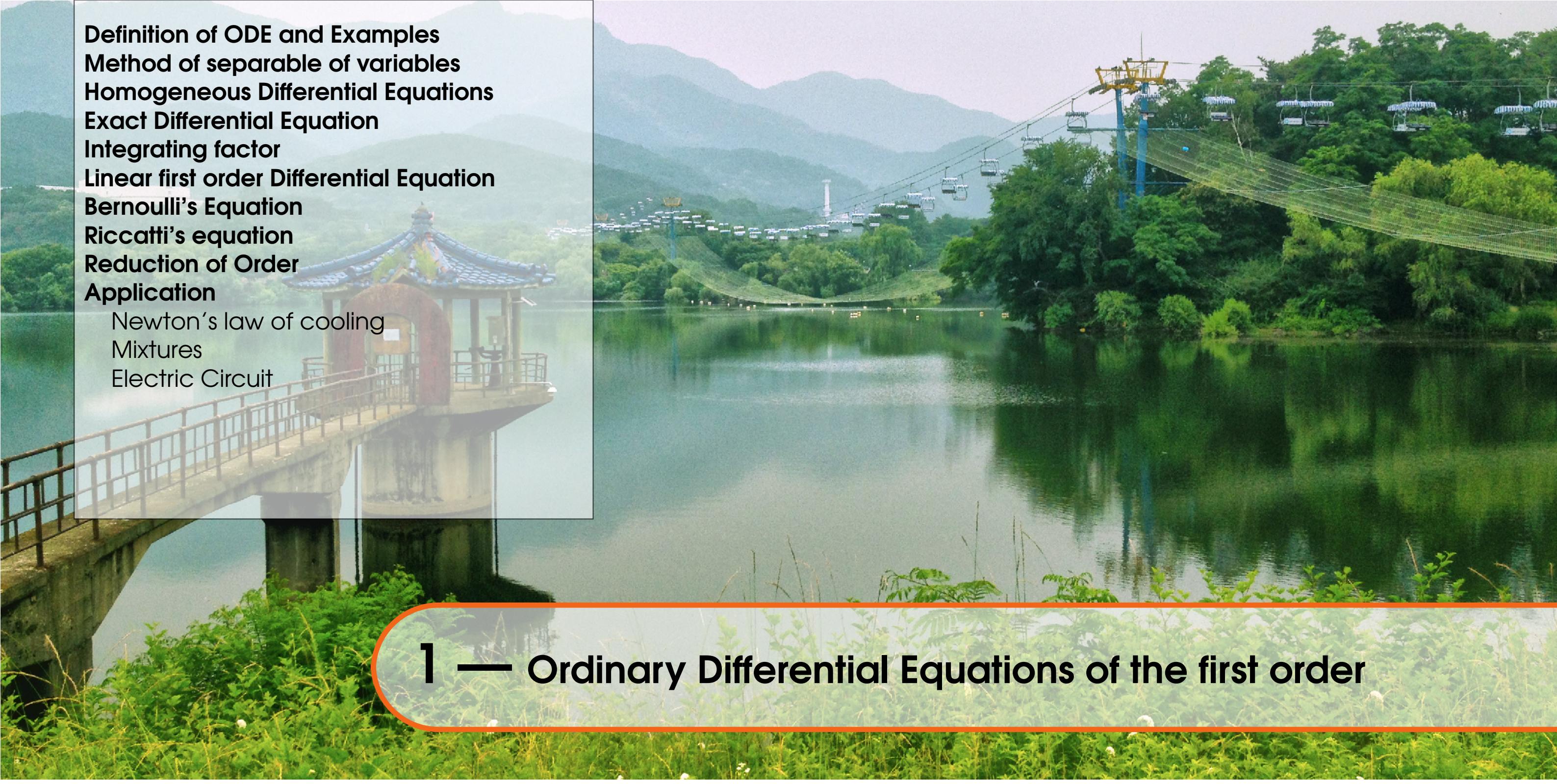

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1 — Ordinary Differential Equations of the first order

1.1 Definition of ODE and Examples

Definition 1.1.1 A differential equation is an equation which involves derivatives.

■ **Example 1.1**

$$a)\ \frac{dy}{dx}=2x+5 \quad b)\ x\frac{d^2y}{dx^2}+6\frac{dy}{dx}=-2y \quad c)\ \frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}-z=0 \quad d)\ x\frac{dy}{dx}+y=3$$

$$e)\ \frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}=\cos x \quad f)\ \left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^4+9y=x^3 \quad g)\ \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=x^2+y$$

■

A differential equation involving only ordinary derivatives (derivatives of functions of one variable) is called *ordinary differential equation*.

■ **Example 1.2** Examples a, b, d, e, f are an example of ODE. ■

If there are two or more independent variables, the derivatives are partial derivatives and the equations are called a PDE.

■ **Example 1.3** Examples c,and g are an example of PDE. ■

Order of a differential equation

Definition 1.1.2 The order of a differential equation is defined as the order of the highest derivative which appears in the equation.

■ **Example 1.4** Examples a, c, d are of order one where as b, f, g are of order two and e is of order three ■

Definition 1.1.3 The degree of a differential equation is the highest power of the highest derivative in the equation, after the differential equation is expressed as a polynomial of the dependent variable and its derivatives.

Example 1.5

(a) $y'' - 2xy' + y = e^x$ order 2, degree 1

(b) $(y''')^4 + 5(y'')^5 - 2y' + y = x^2 + 2$ order 3, degree 4

(c) $(y')^{\frac{3}{2}} = y'' + 1$.

First let us write the given differential equation as a polynomial of the dependent variable and its derivatives, that is,

$(y')^{\frac{3}{2}} = y'' + 1 \implies (y')^3 = (y'' + 1)^2 = (y'')^2 + 2y'' + 1.$

Therefore, the degree of the given differential equation is 2.

Definition 1.1.4 A solution of ODE is free from derivatives and which satisfies the given differential equation.

If a solution of a differential equation is given explicitly as $y = f(x)$ we call it an explicit solution, otherwise it is of the form $h(x, y) = 0$ called implicit solution.

Example 1.6 Show that e^{2x} and e^{3x} are solution of $y'' - 5y' + 6y = 0$

Definition 1.1.5 A differential equation is said to be linear if it is linear in the dependent variable and its derivatives, and those coefficients are a function of the independent variable. *That is, a differential equation is linear if the independent variable and its derivatives are not multiplied together, not raised to powers, do not occur as the arguments of functions.*

A differential equation which is not linear in some dependent variable is said to be non linear.

R An n^{th} order differential equation is linear if it can be written of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

where the coefficients $a_i(x)$ are function of x alone.

Example 1.7 For example

	Differential equation	Linearity
1	$y'' + 4xy' + 2y = cosx$	Is linear, ordinary and order 2
2	$y'' + 4yy' + 2y = cosx$	Is nonlinear ($\because yy'$)
3	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial t} + u + v = \sin u$	Is linear in v and nonlinear in u ($\because \sin u$). The equation is nonlinear.

Initial value problem and Boundary Value Problem

In application one may be interested to find a solution to a differential equation satisfying certain defined conditions and such conditions are called initial conditions. If all conditions are given at one point of independent variable the conditions are called initial conditions and if the conditions are given at more than one point of the independent variable the conditions are called boundary conditions.

Definition 1.1.6 An IVP is a problem which seeks to determine a solution to a differential equation on the unknown functions and its derivatives specified at one value of the independent variable.

■ **Example 1.8** Consider the differential equation $\frac{d^2y}{dx^2} = x+1$ subject to the condition $y(0) = 1, y'(0) = 0$. So the given problem is an IVP ■

Definition 1.1.7 A BVP is a problem which seeks to determine a solution to a differential equation subject to the boundary conditions on the unknown functions and its derivatives specified at least at two different values of the independent variable.

■ **Example 1.9** Consider the differential equation $y'' + y = 0$ subject to the condition $y(0) = 0, y'(\frac{\pi}{2}) = 1$. So the given problem is BVP ■

1.2 Method of separable of variables

Definition 1.2.1 A differential equation of the form

$$g(x) + h(y)\frac{dy}{dx} = 0$$

is called separabel equation

The solution is obtained by integrating both sides with respaect to x

$$\int h(y)dy + \int g(x)dx = c$$

is the general solution.

■ **Example 1.10** Solve the following differential equations by separation of variables

1. $\frac{dy}{dx} = \frac{x^2}{y}$
2. $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$
3. $\frac{dy}{dx} = y - y^2$
4. $\frac{dy}{dx} = 1 + y^2 - 2x - 2xy^2, \quad y(0) = 0$

■

Solution:

1. The ODE $\frac{dy}{dx} = \frac{x^2}{y}$ becomes $ydy = x^2dx$

$$\Rightarrow \int ydy = \int x^2dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + c$$

$$\Rightarrow y = \sqrt{\frac{2x^3}{3} + c}$$

2. The ODE $\dfrac{dy}{dx} = \dfrac{x^2}{y(1+x^3)}$ becomes $ydy = \dfrac{x^2}{1+x^3}dx$

$$\Rightarrow \int ydy = \int \frac{x^2}{1+x^3}dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{1}{3}ln(1+x^3)+c$$

$$\Rightarrow y = \sqrt{\frac{2}{3}\ln(1+x^3)+c}$$

3. The ODE $\dfrac{dy}{dx} = y-y^2$ can be written as $\dfrac{1}{y-y^2}dy = dx$

$$\Longrightarrow \int \frac{1}{y-y^2}dy = \int dx \qquad \text{(Integrating both side w.r.t. } x\text{)}$$

$$\Longrightarrow \int \frac{1}{y(1-y)}dy = \int dx$$

$$\Longrightarrow \int \left(\frac{1}{y} - \frac{1}{y-1}\right)dy = \int dx \qquad \text{(integration by partial fraction)}$$

$$\Longrightarrow \ln(y)-\ln(y-1) = x+c$$

$$\Longrightarrow \ln\frac{y}{y-1} = x+c \Longrightarrow \frac{y}{y-1} = e^{x+c}$$

$$\Longrightarrow \frac{y}{y-1} = Ce^x \qquad \text{(where } C = e^c\text{)}$$

$$\Longrightarrow y = (y-1)Ce^x \Longrightarrow y = \frac{Ce^x}{Ce^x-1}$$

4. The ODE can be written as $\dfrac{dy}{dx} = (1-2x)+y^2(1-2x)$

$$\Longrightarrow \frac{dy}{dx} = (1-2x)(1+y^2)$$

$$\Longrightarrow \frac{1}{1+y^2}dy = (1-2x)dx$$

$$\Longrightarrow \int \frac{1}{1+y^2}dy = \int (1-2x)dx \qquad \text{(integration by trignometric substitution let } y = \tan\theta\text{)}$$

$$\Longrightarrow \tan^{-1}y = x-x^2+c$$

$$\Longrightarrow y = \tan(x-x^2+c) \qquad \text{(The general solution)}$$

From the initial condition $y(0) = 0$, we obtain,

$$y(0) = \tan(0-0+c) \Longrightarrow 0 = \tan c \Longrightarrow c = \tan^{-1}0 = 0$$

Thus, the solution of the IVP is

$$y = \tan(x-x^2)$$

Exercise 1.1 Solve

(a) $y' + y^2\sin x = 0$

(b) $y' = e^{x+y}$

(c) $\dfrac{2y}{y^2+1}\dfrac{dy}{dx} = \dfrac{1}{x^2}$

(d) $x^2y^2dx-(1+x^2)dy = 0, \quad y(0) = 1$

(e) $(e^{2y}-y)\cos x\dfrac{dy}{dx} = e^y\sin 2x, \quad y(0) = 0$

(f) $(xy+2x+y+2)dx+(x^2+2x)dy = 0$

(g) $(x\cos y+(x^2-1)\sin y\dfrac{dy}{dx} = 0, \quad y(0) = \frac{\pi}{3}$

■

1.3 Homogeneous Differential Equations

Definition 1.3.1 A function $f(x,y)$ is called homogeneous of degree n if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$

Example 1.11 a $f(x,y)=x^5-x^2y^3$ is homogeneous of degree 5.
b $f(x,y)=x^3+\sin x\cos y$ is not homogeneous because $f(\lambda x,\lambda y)\neq\lambda^n f(x,y)$.
c $f(x,y)=e^{\frac{y}{x}}+\tan\frac{y}{x}$ is homogeneous of degree 0. ■

Definition 1.3.2 The differential equation of the form

$$M(x,y)dx+N(x,y)dy=0 \tag{1.1}$$

is said to be homogeneous iff $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Equation (1.1) can be written in the form

$$\frac{dy}{dx}=f(x,y) \text{ where } f(x,y)=-\frac{M(x,y)}{N(x,y)}$$

which is homogeneous of degree 0.

$$\begin{aligned}
&\Rightarrow\quad f(\lambda x,\lambda y)=\lambda^0 f(x,y)=f(x,y),\quad \text{set } \lambda=\frac{1}{x}\\
&\Rightarrow\quad f(x,y)=f(1,\frac{y}{x})=f(1,z),\ \ z=\frac{y}{x}\\
&\Rightarrow\quad y=zx\Rightarrow\frac{dy}{dx}=z+x\frac{dz}{dx}\ \ \Rightarrow\ \ z+x\frac{dz}{dx}=f(1,z)\\
&\Rightarrow\quad \frac{xdz}{f(1,z)-z}=dx\ \ \Rightarrow\ \ \frac{dz}{f(1,z)-z}-\frac{dx}{x}=0\ \ \text{(which is separable)}
\end{aligned}$$

Example 1.12 Solve

$$(y^2+2xy)dx-x^2dy=0$$ ■

Solution: Both terms $(M(x,y)=y^2+2xy,\ \ \&\ \ N(x,y)=-x^2)$ in the differential equation are homogeneous of degree 2, so the equation itself is homogeneous. Differentiating the substitution $y=zx$ gives

$$\frac{dy}{dx}=z+x\frac{dz}{dx}\ \ \Rightarrow\ \ dy=zdx+xdz$$

The given differential equation $(y^2+2xy)dx-x^2dy=0$ becomes $\frac{dy}{dx}=\frac{y^2+2xy}{x^2}=\frac{y^2}{x^2}+\frac{2y}{x}$
Substituting $y=zx$ and $\frac{dy}{dx}$ in the differential equation we obtain the variables separable equation,

$$\begin{aligned}
z+x\frac{dz}{dx}=z^2+2z\ \ &\Rightarrow\ \ \frac{dz}{z^2+z}=\frac{dx}{x}\ \ \Rightarrow\ \ \int\frac{dz}{z^2+z}=\int\frac{dx}{x}\\
&\Rightarrow\ \ \int\frac{dz}{z(z+1)}=\ln x+\ln c\Rightarrow\ln z-\ln(z+1)=\ln cx\\
&\Rightarrow\ \ \ln\frac{z}{z+1}=\ln cx\ \ \Rightarrow\ \ \frac{z}{z+1}=cx\ \ \Rightarrow\ \ \frac{\frac{y}{x}}{\frac{y}{x}+1}\ \ \Rightarrow\ \ \frac{y}{x+y}=cx
\end{aligned}$$

Therefore the general solution of the given differential equation is

$$y = \frac{cx^2}{1-cx}$$

where C is an arbitrary constant. In this case the general solution is simple and y is determined explicitly in terms of x.

Example 1.13 Solve

$$y' = \frac{y-x}{y+x}$$

Solution: Let $f(x,y) = \dfrac{y-x}{y+x}$

$$f(\lambda x, \lambda y) = \frac{\lambda y - \lambda x}{\lambda y + \lambda x} = \frac{y-x}{y+x} = \lambda^0 f(x,y)$$

Thus, $f(x,y)$ is homogeneous function of degree 0, so the given differential equation itself is homogeneous. Differentiating the substitution y = zx gives

$$\frac{dy}{dx} = z + x\frac{dz}{dx} \quad \Rightarrow \quad dy = zdx + xdz$$

Substituting $y = zx$ and $\dfrac{dy}{dx}$ in the differential equation we obtain ,

$$\begin{aligned}
z + x\frac{dz}{dx} = \frac{zx-x}{zx+x} &\Longrightarrow z + x\frac{dz}{dx} = \frac{z-1}{z+1} \Longrightarrow x\frac{dz}{dx} = \frac{z-1}{1+z} - z \\
&\Longrightarrow x\frac{dz}{dx} = \frac{-z^2-1}{z+1} \\
&\Longrightarrow \frac{z+1}{-z^2-1}dz = \frac{1}{x}dx \Longrightarrow \int \frac{z+1}{-z^2-1}dz = \int \frac{dx}{x} \\
&\Longrightarrow \int \left(-\frac{z}{(z^2+1)} - \frac{1}{z^2+1} \right) dz = \int \frac{dx}{x} \\
&\Longrightarrow -\frac{1}{2}\ln(z^2+1) - \tan^{-1}(z) = \ln x + c \\
&\Longrightarrow \ln(z^2+1) + 2\tan^{-1}(z) = -2\ln x + C \\
&\Longrightarrow \ln\left(\left(\frac{y}{x}\right)^2 + 1 \right) + 2\tan^{-1}\left(\frac{y}{x}\right) = -2\ln x + C \\
&\Longrightarrow \ln\left(\frac{x^2+y^2}{x^2} \right) + 2\tan^{-1}\left(\frac{y}{x}\right) = -\ln x^2 + C \\
&\Longrightarrow \ln(x^2+y^2) - \ln x^2 + 2\tan^{-1}\left(\frac{y}{x}\right) = -\ln x^2 + C \\
&\Longrightarrow \ln(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = C \qquad \text{(implicit solution)}
\end{aligned}$$

Exercise 1.2 Solve

(a) $x^2ydx - (x^3+y^3)dy = 0$

(b) $xy' = 2x+3y$

(c) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$

(d) $(y^2-x^2)dx + xydy = 0$

(e) $y^2dx - (x^2+xy)dy = 0$

(f) $\dfrac{-1}{y}\sin\dfrac{x}{y}dx + \dfrac{x}{y^2}\sin\dfrac{x}{y}dy = 0$

1.4 Exact Differential Equation

Definition 1.4.1 A differential equation of the form

$$M(x,y)dx+N(x,y)dy=0$$

is said to be exact if there is a function f such that $M(x,y)=\dfrac{\partial f}{\partial x}$ and $N(x,y)=\dfrac{\partial f}{\partial y}$

$$\Rightarrow \quad Mdx+Ndy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{d}{dx}f(x,y)=0$$

$\therefore \quad f(x,y)=c$ is the general solution.

R The partial derivatives of M and N exist and continuous.

Theorem 1.4.1 The differential equation

$$M(x,y)dx+N(x,y)dy=0$$

is exact if and only if $\dfrac{\partial N}{\partial x}=\dfrac{\partial M}{\partial y}$

Proof. ($\Longrightarrow$) Assume the given differential equation is exact. By definition, there exist a a differentiable function $f(x,y)$ such that

$$M(x,y)=\frac{\partial f}{\partial x} \qquad \text{and} \qquad N(x,y)=\frac{\partial f}{\partial y}$$

Therefore,

$$\frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial N}{\partial x}$$

The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of$M(x,y)$ and $N(x,y)$.

($\Longleftarrow$) To prove the converse of the theorem, we assume that $\dfrac{\partial N}{\partial x}=\dfrac{\partial M}{\partial y}$. We need to show that there is a function f such that

$$M(x,y)=\frac{\partial f}{\partial x} \qquad \text{and} \tag{1.2}$$

$$N(x,y)=\frac{\partial f}{\partial y} \tag{1.3}$$

Integrating equation (1.2) with respect to x, holding y fixed (this is a partial integration) to obtain

$$f(x,y)=\int M(x,y)dx+h(y) \tag{1.4}$$

where $h(y)$ is an arbitrary function of y (this is the integration "constant" that we must allow to depend on y, since we held y fixed in performing the integration).
Differentiating (1.4) partially with respect to y yields

$$N=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\int M(x,y)dx+\frac{dh}{dy}$$

$$\frac{dh}{dy}=N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx \tag{1.5}$$

To see that there is such a function of y, it suffices to show that the right-hand side in (1.5) is a function of y. We can find $g(y)$ by integrating with respect to y. Because the right hand side in (1.5) is defined on a rectangle, and hence on an interval as a function of x, it suffices to show that the derivative with respect to x is identically zero. But

$$\begin{aligned}\frac{\partial}{\partial x}\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right) &= \frac{\partial N}{\partial x}-\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)dx \\ &= \frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)dx \\ &= \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=0\end{aligned}$$

by hypothesis. so we can find desired function $h(y)$ by integrating eq. (1.5) with respect to y.

$$h(y)=\int\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right)dy$$

Substituting this in Eq. (1.4), $f(x,y)$ becomes

$$f(x,y)=\int M(x,y)dx+\int\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right)dy \tag{1.6}$$

as the desired function with $\frac{\partial f}{\partial x}=M$ and $\frac{\partial f}{\partial y}=N$ ■

■ **Example 1.14** Test the differential equation
a $(9x^2+y-1)dx-(4y-x)dy=0$
b $e^y dx+(xe^y+2y)dy=0$
for exactness and solve it if it is exact. ■

Solution:a) $M(x,y)=9x^2+y-1,\ \ N(x,y)=-(4y-x)=x-4y$
$\Rightarrow\ \frac{\partial N}{\partial x}=1=\frac{\partial M}{\partial y}$
Hence the given differential equation is exact.
By definition there exist a function f of two variables such that

$$\frac{\partial f}{\partial x} = 9x^2+y-1 \tag{1.7}$$
$$\frac{\partial f}{\partial y} = x-4y \tag{1.8}$$

Integrating equation (1.7) with respect to x we get,

$$\begin{aligned}f(x,y) &= \int(9x^2+y-1)dx \\ &= 3x^3+xy+h(y)\end{aligned}$$

where $h(y)$ is constant with respect to x. Differentiate w.r.t y, we obtain

$$\frac{\partial f(x,y)}{\partial y}=x+h'(y)$$

comparing this with equation (1.8), we have

$$x+h'(y)=x-4y\ \Rightarrow\ h'(y)=-4y\ \Rightarrow\ h(y)=-2y^2$$

$\therefore\ \ f(x,y)=3x^2+yx-x-2y^2=c$ is the general solution of the given ODE.
b) Ans. $f(x,y)=xe^y+y^2=c$

Example 1.15 Solve $(x+\tan^{-1}y)dx+\left(\frac{x+y}{1+y^2}\right)dy=0$ ■

Solution: Here we have, $M(x,y)=x+\tan^{-1}y,\;\; N(x,y)=\dfrac{x+y}{1+y^2}$

$$\Rightarrow \frac{\partial N}{\partial x}=\frac{1}{1+y^2}=\frac{\partial M}{\partial y}$$

Hence the given differential equation is exact.
By definition there exist a function f of two variables such that

$$\frac{\partial f}{\partial x} = x+\tan^{-1}y \qquad (1.9)$$

$$\frac{\partial f}{\partial y} = \frac{x+y}{1+y^2} \qquad (1.10)$$

Integrating equation (1.9) with respect to x we get,

$$f(x,y) = \int (x+\tan^{-1}y)dx+h(y)=\frac{1}{2}x^2+x\tan^{-1}y+h(y)$$

where $h(y)$ is constant with respect to x. Differentiate w.r.t y, we obtain

$$\frac{\partial f(x,y)}{\partial y}=\frac{x}{1+y^2}+h'(y)$$

comparing this with equation (1.10), we get

$$\frac{x}{1+y^2}+h'(y)=\frac{x+y}{1+y^2} \implies h'(y)=\frac{y}{1+y^2}$$

$$\implies h(y)=\int\frac{y}{1+y^2}dy=\frac{1}{2}\ln(1+y^2)$$

$$\therefore\; f(x,y)=\frac{1}{2}x^2+x\tan^{-1}y+\frac{1}{2}\ln(1+y^2)$$

The general solution of the given ODE is

$$\frac{1}{2}x^2+x\tan^{-1}y+\frac{1}{2}\ln(1+y^2)=c$$

Exercise 1.3 Determine which of the following equations are exact and solve it, if it is exact

(a) $(e^x\sin y-2y\sin x)dx+(e^x\cos y+2\cos x)dy=0$
(b) $(y-x^3)dx+(x+y^3)dy=0$
(c) $(\sin x\sin y-xe^y)dy=(e^y+\cos x\cos y)dx$
(d) $dx=\dfrac{y}{1-x^2y^2}dx+\dfrac{x}{1-x^2y^2}dy$
(e) $\dfrac{-1}{y}\sin\dfrac{x}{y}dx+\dfrac{x}{y^2}\sin\dfrac{x}{y}dy=0$
(f) $(3x^2+y^2)dx+2xydy=0$
(g) $3x^2ydx+x^3dy=0$
(h) $2x\sin ydx+x^2\cos ydy=0$
(i) $(6xy-y^3)dx+(4y+3x^2-3xy^2)dy=0$

■

Answer: a) $e^x\sin y+2y\cos x=c$ b) $4xy-x^4+y^4=c$ c) $xe^y+\sin x\sin y=c$
d) $\ln\left(\dfrac{1+xy}{1-xy}\right)-2x=c$ e) $\cos\dfrac{x}{y}=c$ or $\dfrac{x}{y}=c$

1.5 Integrating factor

Definition 1.5.1 Let $M(x,y)dx+N(x,y)dy=0$ is not exact, any function μ which makes $\mu\,(M(x,y)dx+N(x,y)dy)=0$ exact is called integrating factor.

Example 1.16 Show that $\mu = ye^x$ is an integrating factor for the differential equation

$$\left(\frac{\sin y}{y}-2e^{-x}\sin x\right)dx+\left(\frac{\cos y+2e^{-x}\cos x}{y}\right)dy=0$$

and use this fact to find a solution to the differential equation.

Answer: $f(x,y)=e^x\sin y+2y\cos x$

Theorem 1.5.1 A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ has an integrating factor if it has a general solution

Proof. Let $f(x,y)=c$ be the general solution. Then

$$\begin{aligned}
\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0 \quad &\Rightarrow \quad \frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\\
&\Rightarrow \quad \frac{dy}{dx}=-\frac{M}{N}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\\
&\Rightarrow \quad \frac{\frac{\partial f}{\partial y}}{N}=\frac{\frac{\partial f}{\partial x}}{M}=\mu \quad \text{(let)}\\
&\Rightarrow \quad \frac{\partial f}{\partial x}=\mu M \text{ and } \frac{\partial f}{\partial y}=\mu N
\end{aligned}$$

Multiplying the given equation by μ

$$\mu M(x,y)dx+\mu N(x,y)dy=0$$

which is exact. ■

Finding the integrating factor

$$\begin{aligned}
&\mu M(x,y)dx+\mu N(x,y)dy=0\\
\Rightarrow\;& \frac{\partial(\mu N)}{\partial x}=\frac{\partial(\mu M)}{\partial y}\\
\Rightarrow\;& \mu\frac{\partial N}{\partial x}+N\frac{\partial \mu}{\partial x}=\mu\frac{\partial M}{\partial y}+M\frac{\partial \mu}{\partial y}\\
\Rightarrow\;& N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\
\Rightarrow\;& \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}\right)=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} \qquad (1.11)
\end{aligned}$$

Case 1: Let μ be a function of x alone then (1.11) become

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{1}{\mu}\frac{\partial \mu}{\partial x} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = g(x) \quad \Rightarrow \quad \frac{d}{dx}(\ln \mu) = g(x)$$

$$\Rightarrow \quad \ln \mu = \int g(x)dx \quad \Rightarrow \quad \mu = e^{\int g(x)dx}$$

Case 2: Similarly if μ is a function of y alone

$$\frac{1}{\mu}\frac{\partial \mu}{\partial y} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = h(y)$$

$$\Rightarrow \quad \mu = e^{\int h(y)dy}$$

■ **Example 1.17** Solve $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$ ■

Solution: $M(x,y) = 3x^2y + 2xy + y^3$, $\quad N(x,y) = x^2 + y^2$

$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2 \qquad \frac{\partial N}{\partial y} = 2x$

The differential equation is not exact

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2 + 2x + 3y^2 - 2x = 3(x^2 + y^2)$$

$$\Rightarrow \quad g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 3\frac{x^2 + y^2}{x^2 + y^2} = 3$$

$$\mu = e^{\int g(x)dx} = e^{\int 3dx} = e^{3x}$$

Thus the ODE is reduced in to

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0$$

which is exact.

By definition there exist a function f such that

$$\frac{\partial f}{\partial x} = e^{3x}(3x^2y + 2xy + y^3) \text{ and } \frac{\partial f}{\partial y} = e^{3x}(x^2 + y^2)$$

Integrating the second w.r.t. y,

$$f(x,y) = \int e^{3x}(x^2 + y^2)dy = e^{3x}(x^2y + \frac{y^3}{3} + c(x))$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = e^{3x}(2xy) + 3e^{3x}(x^2y + \frac{y^3}{3}) + c'(x)$$

$$= \quad e^{3x}(2xy + 3x^2y + y^3) + c'(x)$$

By comparing the above equation, we have $c'(x) = 0 \quad \Rightarrow \quad c(x) = c$. Hence

$$f(x,y) = e^{3x}(x^2y + \frac{y^3}{3}) = k$$

is the general solution.

Example 1.18 Solve the initial value problem $ydx+(2x-ye^y)dy=0, \quad y(0)=1$

Solution: $M(x,y)=y, \quad N(x,y)=2x-ye^y \Rightarrow \dfrac{\partial M}{\partial y}=1, \quad \dfrac{\partial N}{\partial x}=2 \Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

The given differential equation is not exact.

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=1-2=-1$$

$$h(y)=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=\frac{-1}{y}=\frac{1}{y}$$

$$\therefore \quad \mu=e^{\int h(y)dy}=e^{\int \frac{1}{y}dy}=e^{\ln y}=y$$

Thus the differential equation is reduced in to

$$y^2dx+y(2x-ye^y)dy=0$$

which is exact.

By definition there exist a function f such that

$$\frac{\partial f}{\partial x}=y^2 \text{ and } \frac{\partial f}{\partial y}=y(2x-ye^2)$$

Integrating the first w.r.t. x, we get

$$f(x,y)=\int y^2dx=y^2x+g(y) \implies \frac{\partial f}{\partial y}=2xy+g'(y)$$

Comparing with the second equation, we have

$$g'(y)=-y^2e^y \implies g(y)=e^y(2+2y-y^2)$$
$$\implies f(x,y)=xy^2+e^y(2+2y-y^2)$$

The general solution is $xy^2+e^y(2+2y-y^2)=c$

From the initial conditions, we get $c=3e$ and

$$xy^2+e^y(2+2y-y^2)=3e$$

is the solution of the IVP.

Example 1.19 Solve $ydx+3xdy=0$

Solution: $M(x,y)=y, \quad N(x,y)=3x \implies \dfrac{\partial M}{\partial y}=1, \quad \dfrac{\partial N}{\partial y}=3$

The differential equation is not exact

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=1-3=-2$$

$$\implies \quad g(x)=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-2}{3x}$$

$$\mu=e^{\int g(x)dx}=e^{\int \frac{-2}{3x}dx}=e^{\frac{-2}{3}\ln x}=\frac{1}{(x)^{2/3}}$$

OR

$$h(y) = \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=\frac{-2}{-y}=\frac{2}{y}$$

$$\mu = e^{\int h(y)dy}=e^{\int \frac{2}{y}dy}=e^{2\ln y}=y^2$$

Hence, for the given differential equation we have two integrating factor. This implies that, integrating factor is not unique.
Thus the ODE is reduced in to

$$\frac{y}{(x)^{2/3}}dx+3(x)^{1/3}dy=0 \quad \text{Or} \quad y^3dx+3xy^2dy=0$$

which is exact.
By definition there exist a function f such that

$$\frac{\partial f}{\partial x}=\frac{y}{(x)^{2/3}} \tag{1.12}$$

$$\frac{\partial f}{\partial y}=3(x)^{1/3} \tag{1.13}$$

Integrating equation (1.13) w.r.t. y,

$$f(x,y)=\int 3(x)^{1/3}dy=3yx^{\frac{1}{3}}+h(x)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}=\frac{y}{(x)^{2/3}}+h'(x)$$

By comparing the above equation with (1.12), we have $h'(x)=0 \Rightarrow h(x)=c$. Hence

$f(x,y)=3yx^{\frac{1}{3}}$

Hence, the general solution is given by

$$3yx^{\frac{1}{3}}=c \implies xy^3=C$$

OR $y^3dx+3xy^2dy=0$

By definition there exist a function f such that

$$\frac{\partial f}{\partial x}=y^3 \tag{1.14}$$

$$\frac{\partial f}{\partial y}=3xy^2 \tag{1.15}$$

Integrating equation (1.14) w.r.t. x,

$$f(x,y)=\int y^3dx=xy^3+g(y)$$

$$\Rightarrow \quad \frac{\partial f}{\partial y}=3y^2+g'(y)$$

By comparing the above equation with (1.15), we have $g'(y)=0 \Rightarrow g(y)=c$. Hence

$f(x,y)=xy^3$

Hence, the general solution is given by

$xy^3=C$

Exercise 1.4 Solve the following differential equation by finding an integrating factor

(a) $(3xy+y^2)+(x^2+xy)y'=0$
(b) $(xy-1)dx+(x^2-xy)dy=0$
(c) $y'+2xy=e^{x-x^2}, \quad y(0)=-1$

1.6 Linear first order Differential Equation

The standard form of a first order differential equation is

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1.16}$$

where p(x) and q(x) are any function of x.

$$\Rightarrow \quad (p(x)y - q(x))dx + dy = 0 \tag{1.17}$$

$$\Rightarrow \quad M(x,y) = p(x)y - q(x), \quad N(x,y) = 1$$

Equation (1.17) is not exact, exactness would require $M_y = N_x$

$$\frac{\partial M}{y} - \frac{\partial N}{x} = p(x)$$

$$\Rightarrow \quad \frac{\frac{\partial M}{y} - \frac{\partial N}{x}}{N} = \frac{p(x)}{1} = p(x)$$

$$\Rightarrow \quad \mu = e^{\int p(x)dx}$$

Multiplying equation (1.16) by μ

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x) = e^{\int p(x)dx}q(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(ye^{\int p(x)dx}\right) = e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x) = e^{\int p(x)dx}q(x)$$

$$\Rightarrow \quad ye^{\int p(x)dx} = \int e^{\int p(x)dx}q(x)dx + c$$

$$\therefore \quad y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + c\right] \text{ (The general solution)}$$

Example 1.20 Solve the following differential equation

a $y' + y\cot x = \sin x$

b $y' + 2xy = 4x$

c $xy' - y = x^2e^{-x}$

Solution:

a $y' + y\cot x = \sin x$ is a linear first order differential equation with $p(x) = \cot x \& q(x) = \sin x$

$$\begin{aligned}
y &= e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + c\right] \\
&= e^{-\int \cot x dx}\left[\int e^{\int \cot x dx}\sin x dx + c\right] \\
&= e^{-\ln\sin x}\left[\int e^{\ln\sin x}\sin x dx + c\right] \\
&= \frac{1}{\sin x}\left[\int \sin^2 x dx + c\right] \\
&= \frac{1}{\sin x}\left[\int \frac{1-\cos 2x}{2}dx + c\right] \\
&= \frac{1}{\sin x}\left[\frac{1}{2x} - \frac{\sin 2x}{4} + c\right] = \frac{1}{\sin x}\left[\frac{1}{2x} - \frac{2\sin x\cos x}{4} + c\right] \\
&= \frac{x}{2\sin x} - \frac{\cos x}{2} + \frac{c}{\sin x}
\end{aligned}$$

b $\dfrac{dy}{dx}+2xy=4x,\ \ p(x)=2x,\ \ q(x)=4x$

$$\begin{aligned} y &= e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx+c\right] \\ &= e^{-\int 2xdx}\left[\int e^{\int 2xdx}4xdx+c\right] \ = \ e^{-x^2}\left[\int e^{x^2}4xdx+c\right] \\ &= e^{-x^2}\left[2e^{x^2}+c\right]=2+ce^{-x^2} \end{aligned}$$

c $xy'-y=x^2e^{-x} \ \Rightarrow \ y'-\dfrac{1}{x}=xe^{-x},\ \ p(x)=-\dfrac{1}{x},\ \ q(x)=xe^{-x}$

$$\begin{aligned} y &= e^{-\int -\frac{1}{x}dx}\left[\int e^{\int -\frac{1}{x}dx}xe^{-x}dx+c\right] \\ &= e^{\ln x}\left[\int e^{-\ln x}xe^{-x}dx+c\right] \ = \ x\left[\int e^{-x}dx+c\right] \\ &= x[-e^{-x}+c]=-xe^{-x}+cx \end{aligned}$$

1.7 Bernoulli's Equation

The differential equation

$$\frac{dy}{dx}+p(x)y=q(x)y^n \tag{1.18}$$

where n is any real number, is called **Bernoulli's equation**. Note that for $n=0$ and $n=1$, equation (1.18) is linear. For $n\neq 0$ and $n\neq 1$ the substitution $z=y^{1-n}$ reduces any equation of form (1.18) to a linear equation.

$$\Longrightarrow \quad \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx}=\frac{y^n}{(1-n)}\frac{dz}{dx}$$

Substitute in equation (**??**), we get

$$\begin{aligned} &\Longrightarrow \quad \frac{y^n}{(1-n)}\frac{dz}{dx}+p(x)y=q(x)y^n \\ &\Longrightarrow \quad \frac{1}{(1-n)}\frac{dz}{dx}+p(x)y^{1-n}=q(x) \quad \text{(Divide both sides by } y^n\text{)} \\ &\Longrightarrow \quad \frac{1}{(1-n)}\frac{dz}{dx}+p(x)z=q(x) \\ &\Longrightarrow \quad \frac{dz}{dx}+(1-n)p(x)z=(1-n)q(x) \quad \text{(Linear first order differential equation)} \end{aligned}$$

The general solution of the Bernoulli equation is

$$y^{1-n}e^{\int(1-n)p(x)dx}=\int(1-n)q(x)e^{\int(1-n)p(x)dx}dx+c$$

■ **Example 1.21** Solve $x\dfrac{dy}{dx}+y=x^2y^2$ ■

Solution: We first rewrite the equation as

$$\frac{dy}{dx}+\frac{1}{x}y=xy^2$$

by dividing by x. With $n = 2$ we have $z = y^{-1}$. We then substitute

$$\frac{dy}{dx} = -y^2 \frac{dz}{dx}$$

into the given equation and simplify. The result is

$$\begin{aligned} & -y^2\frac{dz}{dx} + \frac{1}{x}y = xy^2 \implies \frac{dz}{dx} - \frac{1}{x}y^{-1} = -x \\ \implies & \frac{dz}{dx} - \frac{1}{x}z = -x \implies P(x) = -\frac{1}{x}, Q(x) = -x \end{aligned}$$

Hence,

$$\begin{aligned} z &= e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx + c\right] \\ &= e^{-\int -\frac{1}{x}dx}\left[\int e^{\int -\frac{1}{x}dx}(-x)dx + c\right] = e^{\ln x}\left[\int -xe^{-\ln x}dx + c\right] \\ &= x\left[-\int dx + c\right] = x[-x+c] = -x^2 + cx \\ \implies y^{-1} &= -x^2 + cx \\ \implies y &= \frac{1}{-x^2 + cx} \end{aligned}$$

Exercise 1.5 Solve

(a) $x\frac{dy}{dx} + y = x^4y^3$ (b) $3y^2\frac{dy}{dx} + xy^3 = x$ (c) $y' + xy = xe^{-x^2}y^{-3}$

■

1.8 Riccatti's equation

A differential equation of the form

$$\frac{dy}{dx} + p(x)y + q(x)y^2 = r(x) \tag{1.19}$$

is a Riccatti differential equation.

- If $q(x) = 0$, then (1.19) is first order linear differential equation.
- If $r(x) = 0$, then (1.19) is Bernoulli's differential equation

Riccati differential equation can be solved if at least one non-trivial particular solution is known.

Suppose that $u = u(x)$ is a solution of (1.19) and make the change of variables $y = u + v$ to reduce the Riccati equation into Bernoulli equation. Then $y' = u' + v'$ and the differential equation (1.19) becomes

$$\begin{aligned} & u' + v' + p(x)(u+v) + q(x)(u+v)^2 = r(x) \\ \implies & u' + v' + p(x)(u+v) + q(x)\left(u^2 + 2uv + v^2\right) = r(x) \\ \implies & u' + p(x)u + q(x)u^2 + v' + p(x)v + q(x)v^2 + 2uvq(x) = r(x) \\ \implies & v' + (p(x) + 2q(x)u)v + q(x)v^2 = 0 \quad \text{Since } u' + p(x)u + q(x)u^2 = r(x) \\ \implies & v' + (p(x) + 2q(x)u)v = -q(x)v^2 \quad \textbf{(Bernoulli's differential equation)} \end{aligned}$$

Example 1.22 Solve $y' + \frac{1}{x}y - y^2 = -\frac{4}{x^2}$ with $u = \frac{2}{x}$ a given solution.

Solution: The given differential equation is Riccati's differential equation.

Let $y = \frac{2}{x} + v$ is the general solution. Then $y' = -\frac{2}{x^2} + v'$. Substituting into the given differential equation:

$$-\frac{2}{x^2} + v' + \frac{1}{x}\left(\frac{2}{x} + v\right) - \left(\frac{2}{x} + v\right)^2 = -\frac{4}{x^2}$$
$$-\frac{2}{x^2} + v' + \frac{2}{x^2} + \frac{1}{x}v - \frac{4}{x^2} - \frac{4}{x}v - v^2 = -\frac{4}{x^2}$$
$$v' - \frac{3}{x}v = v^2 \qquad \text{Bernoulli equation with } n = 2$$

Let $z = v^{-1}$. Then, $\frac{dz}{dx} = -v^{-2}\frac{dv}{dx} \implies \frac{dv}{dx} = -v^2\frac{dz}{dx}$

Substituting:

$$v' - \frac{3}{x}v = v^2 \implies -v^2\frac{dz}{dx} - \frac{3}{x}v = v^2$$
$$\frac{dz}{dx} + \frac{3}{x}\frac{1}{v} = -1 \implies \frac{dz}{dx} + \frac{3}{x}z = -1$$
$$\implies \quad z = e^{-\int \frac{3}{x}dx}\left[\int e^{\int \frac{3}{x}dx}(-1)dx + c\right]$$
$$\implies \quad = e^{-3\ln x}\left[-\int e^{3\ln x}dx + c\right]$$
$$\implies \quad v^{-1} = \frac{1}{x^3}\left[-\int x^3 + c\right] = \frac{1}{x^3}\left[\frac{1}{4}x^4 + c\right]$$
$$\implies \quad \frac{1}{v} = \frac{1}{4}x + \frac{c}{x^3} = \frac{x^4 + c_1}{4x^3}$$
$$\implies \quad v = \frac{4x^3}{x^4 + c_1}$$

Therefore, the general solution for the given Riccati equation is

$$y = \frac{2}{x} + \frac{4x^3}{x^4 + c_1}$$

1.9 Reduction of Order

Some differential equation of the second order can be solved by reducing to a first order differential equation.

The general second order differential equation has the from

$$F(x, y, y', y'') = 0$$

To solve we consider two special cases

i **Dependent variable missing**

$$f(x, y', y'') = 0$$

Let $y' = p$ and $y'' = \frac{dp}{dx}$. Then

$$f\left(x, p, \frac{dp}{dx}\right) = 0 \rightarrow \text{(reduced to first order ODE in } p\text{)}$$

Example 1.23 Solve $xy'' - y' = 3x^2$

Solution: The differential equation reduced to

$$x\frac{dp}{dx} - p = 3x^2 \implies \frac{dp}{dx} - \frac{1}{x}p = 3x$$
$$\implies p = y' = 3x^2 + c_1 x \implies y = x^3 + \frac{1}{2}c_1 x^2 + c_2$$

ii **Independent variable missing**

$$g(y,\ y',\ y'') = 0$$

Let $y' = p$ and $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$. Then

$$g\left(y, p, p\frac{dp}{dy}\right) = 0 \rightarrow \text{(reduced to first order ODE in p)}$$

Example 1.24 Solve $2yy'' - (y')^2 = 1$

Solution: The given differential equation reduced to

$$2yp\frac{dp}{dy} - p^2 = 1 \implies 2yp\frac{dp}{dy} = p^2 + 1$$
$$\implies \frac{2p}{p^2+1}dp = \frac{1}{y}dy \implies \int \frac{2p}{p^2+1}dp = \int \frac{1}{y}dy$$
$$\implies p = \sqrt{c_1 y - 1} = \frac{dy}{dx} \implies dx = \frac{1}{\sqrt{c_1 y - 1}}dy$$
$$\implies y = \frac{1}{2}c_1 x\sqrt{c_1 y - 1} + c$$

Example 1.25 Solve $y'' + k^2 y = 0$ k is constant

Solution: The differential equation reduced to

$$p\frac{dp}{dy} + k^2 y = 0 \implies pdp + k^2 ydy = 0$$
$$\implies p^2 + k^2 y^2 = k^2 a^2 \implies p = y' = \pm k\sqrt{a^2 - y^2}$$
$$\implies \frac{dy}{\sqrt{a^2 - y^2}} = \pm kdx \implies \sin^{-1}\frac{y}{a} = \pm kx + b$$
$$\implies y = a\sin(\pm kx + b)$$

The general solution can be $y = c_1 \sin kx + c_2 \cos kx$ (by expanding $\sin(kx + B)$ & changing the from of constant)

Exercise 1.6 Solve the following
(a) $yy'' + (y')^2 = 0$ (b) $y'' - k^2 y = 0$ (c) $(x^2 + 2y')y'' + 2xy' = 0,\ y(0) = 1,\ y'(0) = 0$
d $xy'' = y' + (y')^3$, (e) $yy'' = y^2 y' + (y')^2,\ y(0) = \frac{-1}{2},\ y'(0) = 1$

1.10 Application

1.10.1 Newton's law of cooling

According to Newton's empirical law of cooling, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then

$$\frac{dT}{dt} \propto T - T_m \quad or \quad \frac{dT}{dt} = k(T - T_m)$$

where k is a constant of proportionality.

■ **Example 1.26** A pot of liquid is put on the stove to boil. The temperature of the liquid reaches 170^0F and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is 76^0F. After two minutes the temperature of the liquid in the pot is 123^0F. How long before the temperature of the liquid in the pot will be 84^0F? ■

Solution: $T_m = 76, \ \frac{dT}{dt} = k(T-76), \ T(0) = 170.$
Solving this separable differential equation, we get

$$\begin{aligned}
& \frac{dT}{T-76} = kdt \ \Rightarrow \ \int \frac{dT}{T-76} = \int kdt \ \Rightarrow \ \ln(T-76) = kt + c_1 \\
\Rightarrow \ & T - 76 = Ce^{kt} \ \Rightarrow \ T(t) = Ce^{kt} + 76 \ \Rightarrow \ T(0) = 170 \ \Rightarrow \ C = 170 - 76 = 94 \\
& T(2) = 123 \ \Rightarrow \ 123 = 94e^{2k} + 76 \ \Rightarrow \ 47 = 94^{2k} \ \Rightarrow \ k = \frac{1}{2}\ln\frac{1}{2} = -0.3466 \\
\therefore \ & T(t) = 94e^{-0.3466t} + 76 \\
\Rightarrow \ & 84 = 94e^{-0.3466t} + 76 \ \Rightarrow \ 8 = 94e^{-0.3466t} \ \Rightarrow \ -0.3466t = -2.4639 \ \Rightarrow \ t = 7.1088
\end{aligned}$$

When $t = 7.1088$ minutes the temperature of the liquid in the pot is 84^0F

Exercise 1.7

1. An object with temperature 150^0c is placed in a freezer whose temperature is 30^0c assume that newtons law of cooling applies and that the temperature of the freezer remains essentially constant. If this object is cooled to 120^0c after 8 minutes, what will its temperature be after 16 minutes? When will its temperature be 60^0c?
2. A thermometer is removed from a room where the temperature is 70^oF and is taken outside, where the air temperature is 10^oF. After one-half minute the thermometer reads 50^oF. What is the reading of the thermometer at $t = 1$ min? How long will it take for the thermometer to reach 15^oF?
3. The rate at which a body loses temperature at any instant is proportional to the amount by which the temperature of the body exceeds room temperature at the instant. A container of hot liquid is placed in a room of temperature 19^0c and in 8 minutes the liquid cools from 83^0c to 51^0c. How long does it takes for the liquid to cool from 27^0c to 25^0c?

■

1.10.2 Mixtures

Mixing problem occur quite frequently in chemical industry. Mixture problems generally concern a tank, or reservoir, containing a solution of some substance, being filled at a certain rate with another solution of the same substance, instantaneously mixed with the solution in the tank, and at the same time being drained at a certain rate.
The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture.
Let $A(t)$ denotes the amount of substance in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt}) = R_{in} - R_{out}$$

where $R_{in} = $ (Flow rate of the liquid entering)(Concentration of salt in it)
$R_{out} = $ (Flow rate of the liquid leaving)(Concentration of salt in it)

$$\text{Concentration of salt in the tank at any time t} = \frac{A(t)}{\text{volume of fluid in the tank at any time}}$$

But the volume of brine at time t is given by

$$\begin{aligned}&(\text{initial volume}) + (\text{net change in volume})\\ &= (\text{initial volume}) + (\text{flow rate entering} - \text{flow rate exit})t\end{aligned}$$

■ **Example 1.27** A large tank holds 300 gallons of brine solution. Salt was entering and leaving the tank;A concentration of 2 lbs/gal is pumped into the tank at a rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time? ■

Solution: $\frac{dA}{dt} = R_{in} - R_{out}$

$$R_{in} = \left(2\frac{lbs}{gal}\right)\left(3\frac{gal}{min}\right) = 6\frac{lbs}{min}$$

Now, since the solution is being pumped out of the tank at the same rate that it is pumped in, the number of gallons of brine in the tank at time t is a constant 300 gallons. Hence the concentration of the salt in the tank as well as in the outflow is $c(t) = \frac{A(t)}{300\ lb/gal}$

$$R_{out} = \left(\frac{A}{300}\frac{lbs}{gal}\right)\left(3\frac{gal}{min}\right) = \frac{A}{100}\frac{lbs}{min}$$

$$\begin{aligned}\frac{dA}{dt} &= 6 - \frac{A}{100}, \quad A(0) = 50\\ \frac{dA}{dt} &+ \frac{A}{100} = 6, \quad p(t) = \frac{1}{100}, \quad q(t) = 6\end{aligned}$$

$$\begin{aligned}A(t) &= 600 + ce^{-t/100}, \quad A(0) = 50,\\ \Rightarrow\ 600 + ce^{-0/100} &= 50 \ \Rightarrow\ c = -550 \times 10^7\end{aligned}$$

Thus the amount of salt in the tank at time t is given by

$$A(t) = 600 - 550e^{-t/100}$$

over a long time the number of pounds of salt in the solution must be 600 lb

Example 1.28 A large tank holds 300 gallons of brine solution with 40 *lbs* of salt. A concentration of 2 *lbs/gal* is pumped in at a rate of 4 *gal/min*. The concentration leaving the tank is pumped out at a rate of 3 *gal/min*. How much salt is in the tank after 12 min?

Solution: $\frac{dA}{dt} = R_{in} - R_{out}$

$$R_{in} = \left(2\frac{lbs}{gal}\right)\left(4\frac{gal}{min}\right) = 8\frac{lbs}{min}$$

$$R_{out} = \left(\frac{A}{300+t}\frac{lbs}{gal}\right)\left(3\frac{gal}{min}\right) = \frac{3A}{300+t}\frac{lbs}{min}$$

$$\frac{dA}{dt} = 8 - \frac{3A}{300+t}, \quad A(0) = 40$$

$$\frac{dA}{dt} + \frac{3}{300+t}A = 8, \quad p(t) = \frac{3}{300+t}, \quad q(t) = 8$$

$$A(t) = 600 + 2t + \frac{c}{(300+t)^3}, \quad A(0) = 40,$$

$$\Rightarrow \quad 600 + \frac{c}{300^3} = 40 \quad \Rightarrow \quad c = -1512 \times 10^7$$

How much salt is in the tank after 12 min?

$$A(12) = 600 + 2(10) - \frac{1512 \times 10^7}{(300+12)^3} \approx 126.12\ lbs \quad \text{of salt}$$

Example 1.29 In an oil refinery, a storage tank contains 2000 *gal* of gasoline that initially has 100 *lb* of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 *lb* of additive per gallon is pumped into the tank at a rate of 40 *gal/min*. The well-mixed solution is pumped out at a rate of 45 *gal/min*. How much of the additive is in the tank 20 min after the pumping process begins?

Solution: Let A be the amount (in pounds) of additive in the tank at time t. We know that $A = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$V(t) = 200 + (40\ gal/min - 45\ gal/min)(t\ min) = (2000 - 5t)\ gal$$

Therefore, $R_{out} = \frac{A(t)}{V(t)} \times (Rate\ out\ flow) = \left(\frac{A(t)}{2000-5t}\right)45,$

$R_{in} = \left(2\ \frac{lb}{gal}\right)\left(40\ \frac{gal}{min}\right) = 80\ \frac{lb}{min}$

The differential equation modeling the mixture process is

$$\frac{dA}{dt} = R_{in} - R_{out} = 80 - \frac{45A}{2000-5t}$$

in pounds per minute.

Thus the general solution is

$$A = 2(2000-5t) + C(2000-5t)^9, \quad A(0) = 100 \Rightarrow C = -\frac{3900}{(2000)^9}$$

$$\Rightarrow \quad A(20) = 1342\ lb$$

Exercise 1.8

1. Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred, and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine solution entering the tank is 0.1 Kg/L, determine when the concentration of salt will reach 0.05 Kg/L.
2. Consider a tank in which 1 g of chlorine is initially present in $100m^3$ of a solution of water and chlorine. A chlorine solution concentrated at $0.03g/m^3$ flows into the tank at a rate of $1m^3/min$, while the uniformly mixed solution exits the tank at $2m^3/min$. At what time is the maximum amount of chlorine present in the tank, and how much is present?

1.10.3 Electric Circuit

A **RL-Series circuit:** Kirchoff's second law states that the sum of the voltage, $V(t)$ drop across the inductor, $L(\frac{dI}{dt})$ and across the resistor RI is the same as the impressed voltage $V(t)$ in the circuit where I is current.

$$V(t) = RI + L\frac{dI}{dt}$$

B **RC-Series circuit:** Kirchoff's second law states that the sum of the voltage, $V(t)$ drop across the capacitor, $\frac{1}{C}q(t)$ and across the resistor RI is the same as the impressed voltage $V(t)$ in the circuit where q is the charge on the capacitor.

$$V(t) = RI + \frac{1}{c}q$$

Example 1.30 An RL-circuit has an electromotive force of 5 volts, a resistor of 50Ω an inductance of 1 Henry and no initial current. Find the current in the circuit at any time.

Solution: $V(t) = RI + L\frac{dI}{dt}$, $V = 5,\ R = 50,\ L = 1$

$50I + \frac{dI}{dt} = 5 \ \Rightarrow\ I(t) = \frac{1}{10} - \frac{1}{10}e^{-50t}$

Example 1.31 A 100-volt electromotive force is applied to an RC series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current i(t).

Solution: $V(t) = RI + \frac{1}{c}q \quad\Longrightarrow\quad V(t) = R\frac{dq}{dt} + \frac{1}{c}q$,
$V = 100,\ R = 200,\ C = 10^{-4}$

$$\Longrightarrow\quad 200\frac{dq}{dt} + \frac{1}{10^{-4}}q = 100 \quad\Longrightarrow\quad \frac{dq}{dt} + 50q = \frac{1}{2}$$

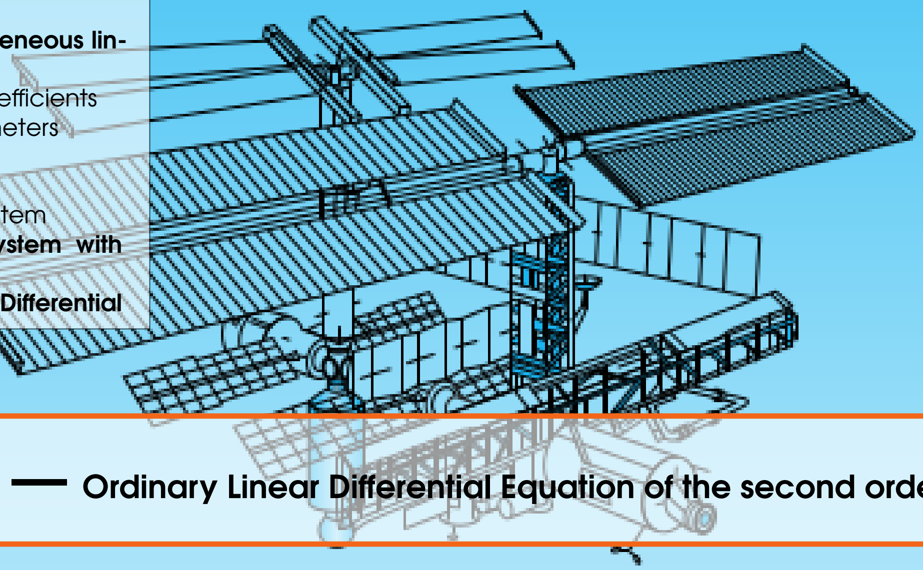
$$\Longrightarrow\quad q = e^{-\int 50dt}\left[\int \frac{1}{2}e^{\int 50dt}dt + c\right] \quad\Longrightarrow\quad q = e^{-50t}\left[\int \frac{1}{2}e^{50t}dt + c\right]$$

$$\Longrightarrow\quad q = e^{-50t}\left[\frac{1}{100}e^{50t} + c\right] \quad\Longrightarrow\quad q = ce^{-50t} + \frac{1}{100}$$

From the initial condition, $q(0) = 0$, we obtain $c = -\frac{1}{100}$. Thus,

$$q = \frac{1}{100} - \frac{1}{100}e^{-50t} \text{ and } I = \frac{dq}{dt} = \frac{1}{2}e^{-50t}$$

Homogeneous Linear Differential Equation of the second order
The use of a known solution to find another (Reduction order)
Homogeneous Differential Equation with constant coefficient
Cauchy-Euler equation
Methods for solving non homogeneous linear differential equations
Method of Undetermined Coefficients
Method of variation of parameters
System of Differential equation
Homogeneous Linear System
Non-homogeneous Linear System
Operator method for Linear System with constant coefficients
Applications of Second-Order Differential Equations
Spring/Mass System
Electric Circuit



2 — Ordinary Linear Differential Equation of the second order

2.1 Homogeneous Linear Differential Equation of the second order

A second-order linear differential equation has the form

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.1}$$

where P , Q and R are continuous functions of x.
If $R(x) = 0$, for all x, then (2.1) reduces

$$y'' + P(x)y' + Q(x)y = 0 \tag{2.2}$$

and is called homogeneous. If $R(x) \neq 0$, , then (2.1) is called non-homogeneous.
Let $y_g(x, c_1, c_2)$ is the general solution of (2.2). Let y_p is a fixed particular solution of (2.1).
If y is any other solution of (2.1) then we can show that $y - y_p$ is a solution of (2.2)

$$\begin{aligned}(y-y_p)'' + P(x)(y-y_p)' + Q(x)(y-y_p) &= y_p'' - y'' + P(x)y_p' - P(x)y' + Q(x)y_p - Q(x)y \\ &= y_p'' + P(x)y_p' + Q(x)y_p - (y'' + P(x)y' + Q(x)y) = R - R = 0\end{aligned}$$

There fore, $y - y_p$ is a solution of $y'' + Py' + Qy = 0$.
Since $y_g(x, c_1, c_2)$ is the general solution of (2.2),

$$\implies \quad y - y_p = y_g(x, c_1, c_2) \quad \implies \quad y = y_p + y_g(x, c_1, c_2)$$

Theorem 2.1.1 — (Principle of supper position) If $y_1(x)$ and $y_2(x)$ are any solution of (2.2) then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (2.2) for any constant c_1 & c_2

Proof. Since y_1 and y_2 are solution of (2.2) we have

$$y_1'' + Py_1' + Qy_1 = 0 \text{ and } y_2'' + Py_2' + Qy_2 = 0$$

Let $y = c_1y_1 + c_2y_2$. We want to show y is solution of (2.2).

$$\begin{aligned}&(c_1y_1 + c_2y_2)'' + P(c_1y_1 + c_2y_2)' + Q(c_1y_1 + c_2y_2) = c_1y_1'' + c_2y_2'' + Pc_1y_1' + Pc_2y_2' + Qc_1y_1 + Qc_2y_2 \\ \Rightarrow\ & c_1(y_1'' + Py_1' + Qy_1) + c_2(y_2'' + Py_2' + Qy_2) \\ \Rightarrow\ & c_1.0 + c_2.0 = 0\end{aligned}$$

Therefore, $c_1y_1 + c_2y_2$ is also a solution of (2.2) ■

R Super position principle in general does't hold for non-homogeneous and non-linear.

Example 2.1 1. $y_1 = 1+\cos x$ and $y_2 = 1+\sin x$ are solutions of the non-homogeneous differential equation $y''+y=1$ but their linear combination $y_1+y_2 = 2+\cos x+\sin x$ is not the solution.

2. $y_1 = x^2$ and $y_2 = 1$ are the solutions of the non-linear DE $yy''-xy'=0$ but their linear combination $y_1+y_2 = x^2+1$ is not the solution.

Linear independence and Wronskian

Definition 2.1.1 If $y_1,\ y_2,\ \dots y_n$ are functions in an interval I and if each function possesses (n-1) derivatives on this interval then the **Wronskian** of the n function is

$$W(x) = W(y_1,\ y_2,\ \dots y_n) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ \vdots & \vdots & \vdots & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

In particular, for two differentiable functions $y_1(x)$ and $y_2(x)$ the Wronskian is defined as

$$W(x) = W(y_1,\ y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{2.3}$$

Definition 2.1.2 A collection of function $\{y_i(x)\}_{i=1}^n$ is linearly independent on (a, b) if $\sum_{i=1}^{n} c_i y_i = 0,\ \forall x \in (a,\ b)$ then $c_i = 0,\ \ (i = 0,\ 1, \dots n)$ otherwise $\{y_i(x)\}_{i=1}^n$ is called linearly dependent.

If $W(y_1,\ y_2) \neq 0$ then the function $y_1(x)$ and $y_2(x)$ are linearly independent and if $W(y_1,\ y_2) = 0$ then they are linearly dependent.

Definition 2.1.3 A set of a linearly independent solutions is called **fundamental set**

Theorem 2.1.2 Let $y_1(x)$ and $y_2(x)$ are linearly independent solution of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{2.4}$$

on the interval [a, b] then $c_1y_1 + c_2y_2$ is the general solution of (2.4).

Corollary 2.1.3 If y_1 and y_2 are any two solution of (2.4) on (a, b) then their Wronskian $W = W(y_1,\ y_2)$ is either identically zero or never zero on [a, b]

Corollary 2.1.4 If y_1 and y_2 are any two solution of (2.4) on (a, b) then they are linearly dependent on this interval if and only if their Wronskian $W = W(y_1,\ y_2) = y_1y_2 - y_2y_1'$ is identically zero.

Example 2.2 Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval. Find the particular solution for which $y(0) = 2$ & $y'(0) = 3$

2.2 The use of a known solution to find another (Reduction order)

Let $y'' + P(x)y' + Q(x)y = 0$ If y_1 and y_2 are linearly independent solution of (2.4), then the general solution is $y = c_1y_1 + c_2y_2$. If y_1 is a solution then cy_1 is also a solution of (2.4). Replace c by a variable v and let $y_2 = vy_1$.
Assume that y_2 is also a solution of (2.4)

$$y_2'' + Py_2' + Qy_2 = 0$$

To find v,

$$\begin{aligned} y_2' = vy_1' + v'y_1 \quad &\text{and} \quad y_2'' = vy_1'' + 2v'y_1' + v''y_1 \\ y_2'' + Py_2' + Qy_2 &= vy_1'' + 2v'y_1' + v''y_1 + P(vy_1' + v'y_1) + Q(vy_1) \\ &= v(y_1'' + Py_1' + Qy_1) + v'(2y_1' + py_1) + v''y_1 \\ &= v''y_1 + v'(2y_1' + py_1) = 0 \\ \Longrightarrow v''y_1 + v'(2y_1' + py_1) = 0 \quad &\Longrightarrow \quad \frac{v''}{v'} = \frac{-2y_1'}{y_1} - P \end{aligned}$$

Integrating

$$\begin{aligned} \ln v' &= -2\ln y_1 - \int P(x)dx \quad \Rightarrow \quad v' = \frac{1}{y_1^2} e^{-\int P(x)dx} \\ \therefore \quad v &= \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx \\ \Rightarrow \quad y_2 &= vy_1 = y_1 \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx \end{aligned}$$

Example 2.3 Let $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution.

Solution: $x^2y'' + xy' - y = 0 \;\Rightarrow\; y'' + \frac{1}{x}y'' - \frac{1}{x^2}y = 0, \quad p(x) = \frac{1}{x}, \quad y_2 = vy_1$

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx = \int \frac{1}{x^2} e^{-\int \frac{1}{x}dx} dx = \int \frac{1}{x^2} e^{-\ln x} dx = \int \frac{1}{x^3} dx = -\frac{1}{2x^2} \\ \therefore \quad y_2 &= vy_1 = -\frac{1}{2x^2}.x = -\frac{1}{2x} \end{aligned}$$

The general solution is $y = c_1x + c_2x^{-1}$

Exercise 2.1 Find the general solution of
(a) $y'' + y = 0, \; y_1 = \sin x$ (b) $y'' - y = 0, \; y_1 = e^x$
(c) $xy'' + 3y' = 0, \; y_1 = 1$ (d) $(1 - x^2)y'' - 2xy' + 2y = 0, \; y_1 = x$

Answer:

a $y = c_1 \sin x + c_2 \cos x$
b $y = c_1e^x + c_2e^{-x}$
c $y = c_1 + c_2x^{-2}$
d $y = c_1x + c_2\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right)$

2.3 Homogeneous Differential Equation with constant coefficient

The special case of $y'' + p(x)y' + q(x)y = 0$ for which $p(x)$ and $q(x)$ are constants

$$y'' + py' + qy = 0 \tag{2.5}$$

Let $y = e^{mx}$ be possible solution of (2.5)
$y' = me^{mx}, \quad y'' = m^2 e^{mx}$

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0 \quad \Rightarrow \quad (m^2 + pm + q)e^{mx} = 0$$
$$\Rightarrow \quad m^2 + pm + q = 0 \quad \rightarrow \text{ This equation is called auxilary/characterstics } equation$$

The two roots m_1 and m_2

$$m_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \quad m_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

Case 1: Distinct two real roots $(p^2 - 4q > 0)$.
We have two solutions $e^{m_1 x}$ and $e^{m_2 x}$. (Let m_1 and m_2 solution for characterstics equation)

$$\frac{e^{m_1 x}}{e^{m_2 x}} = e^{(m_1 - m_2)x} \text{ is not constant} \quad \Rightarrow \quad e^{m_1 x} \text{ and } e^{m_2 x} \text{ are linearly independent.}$$

The general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Case 2: If $p^2 - 4q = 0$ (One solution)
$y = e^{mx}$ is a solution where $m = \dfrac{-p}{2}$
Let $y_1 = e^{-\frac{p}{2}x}$, then $y_2 = vy_1$

$$\Rightarrow \quad y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p dx} dx = e^{-\frac{p}{2}x} \int \frac{1}{y_1^2} e^{-px} dx = e^{-\frac{p}{2}x} \int \frac{1}{e^{-px}} e^{-px} dx = xe^{-\frac{p}{2}x}$$

The general solution is $y = c_1 y_1 + c_2 y_2 \quad \Rightarrow \quad y = c_1 e^{-\frac{p}{2}x} + c_2 x e^{-\frac{p}{2}x}$
Case 3: If $p^2 - 4q < 0$. In this case m_1 and m_2 can be written as $a \pm ib$

$$\begin{aligned} e^{m_1 x} &= e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx), \quad e^{m_2 x} = e^{(a-ib)x} = e^{ax}(\cos bx - i \sin bx) \\ \Rightarrow \quad e^{m_1 x} + e^{m_2 x} &= 2e^{ax} \cos bx, \quad e^{m_1 x} - e^{m_2 x} = 2ie^{ax} \sin bx \\ \therefore \quad y &= e^{ax}(c_1 \cos bx + c_2 \sin bx) \end{aligned}$$

Example 2.4 Solve the following

(a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$ (b) $2y'' - 3y' = 0$ (c) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0$ (d) $\dfrac{d^2y}{dx^2} + y = 0$

Exercise 2.2 1. Find the general solution of

(a) $y'' - 5y' - 14y = 0$
(b) $y'' + 3y' + 3y = 0$
(c) $y'' + 10y' + 25y = 0$
(d) $4y'' - 5y' = 0, y(-2) = 0, y'(-2) = 7$
(e) $y'' + 14y' + 49y = 0, \quad y(-4) = -1, \ y'(-4) = 5$

2.3.1 Cauchy-Euler equation

A linear differential equation of the form

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0 \tag{2.6}$$

where the coefficients a,b,c are constants, is known as a Cauchy-Euler equation.
Let $y=x^m$ be possible solution of (2.6)

$$\begin{aligned} ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy &= ax^2m(m-1)x^{m-2}+bxmx^{m-1}+cx^m=0 \\ &\Longrightarrow (am(m-1)+bm+c)x^m=0 \\ &\Longrightarrow am(m-1)+bm+c=0,\;\; x^m\neq 0 \\ &\Longrightarrow am^2+(b-a)m+c=0 \end{aligned} \tag{2.7}$$

CASE I : ***DISTINCT REAL ROOTS:*** Let m_1 and m_2 denote the real roots of ((2.7)) such that $m_1\neq m_2$. Then $y_1=x^{m_1}$ and $y_2=x^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$y=c_1x^{m_1}+c_2x^{m_2}$$

CASE II : ***REPEATED REAL ROOTS:*** If the roots of (2.7) are repeated (that is, $m_1=m_2$), then we obtain only one solution—namely, yx^{m_1} . When the roots of the quadratic equation $am^2+(b-a)m+c=0$ are equal, the discriminant of the coefficients is necessarily zero. It follows from the quadratic formula that the root must be $m_1=-\dfrac{b-a}{2a}$
Now we can construct a second solution y_2 , using reduction of order. We first write the Cauchy-Euler equation in the standard form

$$\frac{d^2y}{dx^2}+\frac{b}{ax}\frac{dy}{dx}+\frac{c}{ax^2}y=0$$

Hence, $P(x)=\dfrac{b}{ax}\Longrightarrow\displaystyle\int\frac{b}{ax}=\frac{b}{a}\ln x$. Thus,

$$\begin{aligned} y_2 &= y_1\int\frac{1}{y_1^2}e^{-\int P(x)dx}dx=x^{m_1}\int\frac{1}{x^{2m_1}}e^{-\frac{b}{a}\ln x}dx \\ &= x^{m_1}\int x^{-2m_1}x^{-\frac{b}{a}}dx=x^{m_1}\int x^{\frac{b-a}{a}}x^{-\frac{b}{a}}dx \\ &= x^{m_1}\int\frac{1}{x}dx=x^{m_1}\ln x \end{aligned}$$

The general solution is then

$$y=c_1x^{m_1}+c_2x^{m_1}\ln x$$

CASE II : ***CONJUGATE COMPLEX ROOTS:*** If the roots of (2.7) are the conjugate pair

$$m_1=\alpha+i\beta, m_2=\alpha-i\beta,$$

then the general solution is

$$y=x^{\alpha}\left[c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)\right]$$

Example 2.5 Solve
(a) $x^2y''+3xy'+10y=0$
(b) $2x^2y''+10xy'+8y=0$
(c) $x^2y''+2xy'-12y=0$

2.4 Methods for solving non homogeneous linear differential equations

2.4.1 Method of Undetermined Coefficients

Consider

$$y'' + p(x)y' + q(x)y = R(x) \tag{2.8}$$

if $y_g(x)$ (the general solution of the associated homogenous equation) is know and y_p is a particular solution of (2.8) then

$$y = y_g(x) + y_p(x)$$

is the general solution of (2.8).
Now let us see how to found y_p with some special cases where

- the coefficients p and q are constants and
- $R(x)$ is a constant k, a polynomial function, an exponential function e^{ax} , a sine or cosine function $\sin bx$ or $\cos bx$, or finite sums and products of these functions.

The procedure for finding y_p is called the method of undetermined coefficients.

- If $R(x) = e^{ax}$ then take $y_p = Ae^{ax}$, where A is the undetermined coefficients and a is not roots of the auxiliary equation $m^2 + pm + q = 0$.
  Hence, $A = \dfrac{1}{a^2 + pa + q}, \quad a^2 + pa + q \neq 0$
  - If a is a single roots of the auxiliary equation $m^2 + pm + q = 0$, then take $y_p = Axe^{ax}$.
    Thus $A = \dfrac{1}{2a + p}, \quad 2a + p \neq 0$
  - If a is a double roots of the auxiliary equation $m^2 + pm + q = 0$, then take $y_p = Ax^2e^{ax}$.
    Thus $A = \dfrac{1}{2}$
- If $R(x) = \sin bx$ then take $y_p = A\sin bx + B\cos bx$,
  The undetermined coefficients A and B can how be computed by substituting and equating the resulting coefficients of *sinbx* and *cosbx*.
- If $R(x) = a_0 + a_1x + a_x^2 + \ldots + a_nx^n$, take $y_p = A_0 + A_1x + A_1x^2 + \ldots + A_nx^n$.
  If the constant q happens to be zero, then this procedure gives x^{n-1} as the highest power of x on the left of (2.8), so in this case we take our trial solution in the form
  $$y_p = x(A_0 + A_1x + A_1x^2 + \ldots + A_nx^n)$$

R If any y_{pi} contains terms that duplicate terms in y_g , then that y_{pi} must be multiplied by x^n , where n is the smallest positive integer that eliminates that duplication.

Example 2.6 Find the general solution of

a $y'' + 3y' - 10y = 6e^{4x}$
b $y'' + 4y = 3\sin x$
c $y'' - 2y' + 5y = 25x^2 + 12$

Exercise 2.3 Find the general solution of

(a) $y'' - 4y' + 4y = e^{2x}$
(b) $y'' + 4y = 3\cos 2x$
(c) $y'' + 4y = \sin x + \sin 2x$
(d) $y'' + y = 4x + 10\sin x, \quad y(\pi) = 0, \ y'(\pi) = 2$
(e) $y'' + 2y' + 4y = 8x^2 + 12e^{-x}$

(f) $y'' + 2y' + 4y = 8x^2 + 12e^{-x} + 10\sin 3x$

2.4.2 Method of variation of parameters

Techniques for determining a particular solution of the non homogeneous equation

$$y'' + py' + qy = R(x)$$

Let $y = c_1y_1(x) + c_2y_2(x)$ be the general solution of the corresponding homogeneous equations. Now we replace c_1 & c_2 by a known function v_1 & v_2

$$\begin{aligned} y(x) &= v_1y_1 + v_2y_2 \\ y'(x) &= v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2' \\ &= (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2') \end{aligned}$$

Let $v_1'y_1 + v_2'y_2 = 0$

$$\begin{aligned} \Rightarrow \; y' &= v_1y_1' + v_2y_2' \\ y'' &= v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' \\ &= (v_1'y_1' + v_2'y_2') + v_1y_1'' + v_2y_2'' \end{aligned}$$

Substituting y, y', and y'' in the given equation we get

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2'') + v_1'y_1' + v_2'y_2' = R(x)$$

$$\Rightarrow \begin{cases} v_1'y_1' + v_2'y_2' = R(x) \\ v_1'y_1 + v_2'y_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ R(x) \end{pmatrix}$$

$$\Rightarrow \; v_1' = \frac{-y_2R(x)}{W(y_1,y_2)} \;\; \& \;\; v_2' = \frac{y_1R(x)}{W(y_1,y_2)}$$

$$\Rightarrow \; v_1 = \int \frac{-y_2R(x)}{W(y_1,y_2)}dx, \quad v_2 = \int \frac{y_1R(x)}{W(y_1,y_2)}dx$$

$$\therefore \; y_p = y_1 \int \frac{-y_2R(x)}{W(y_1,y_2)}dx + y_2 \int \frac{y_1R(x)}{W(y_1,y_2)}dx$$

Example 2.7 Find the particular solution of $y'' + y = \csc x$

Solution: The corresponding homogeneaous equation is

$$\begin{aligned} & y'' + y = 0 \\ \Rightarrow \; & y_g = c_1\sin x + c_2\cos x \\ \Rightarrow \; & y_1 = \sin x, y_2 = \cos x \Rightarrow \quad W(y_1,y_2) = y_1y_2' - y_2y_1 = -\sin^2 x - \cos^2 x = -1 \end{aligned}$$

$$\begin{aligned} v_1 &= \int \frac{-y_2R(x)}{W(y_1,y_2)}dx \\ &= \int \frac{-\cos x\csc x}{-1}dx = \int \frac{\cos x}{\sin x} = \ln(\sin x) \\ v_2 &= \int \frac{y_1R(x)}{W(y_1,y_2)}dx \\ &= \int \frac{\sin x\csc x}{-1}dx = \int -dx = -x \\ \therefore \quad & y_p = v_1y_1 + v_2y_2 = \sin x\ln(\sin x) - x\cos x \end{aligned}$$

Example 2.8 Find the general solution of

$$y'' + 5y' + 6y = e^{-x}$$

Solution: The characteristics equation of the corresponding homogeneous DE is

$$m^2 + 5m + 6 = 0$$

Then the solution of the corresponding homogeneous equation is

$$y_g = c_1 e^{-3x} + c_2 e^{-2x}$$

Using variation of parameter with $y_1 = e^{-3x}, y_2 = e^{-2x}$ and $W = \begin{vmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{vmatrix} = e^{-5x}$.

Thus, we get

$$y_p = \frac{1}{2}e^{-x}$$

The general solution is:

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{1}{2}e^{-x}$$

Example 2.9 Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$

Solution: Since the equation is non-homogeneous, we first solve the associated homogeneous equation. From the auxiliary equation $(m-1)(m-3) = 0$ we find $y_g = c_1x + c_2x^3$.

The given differential equation can be written in the form

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x.$$

Using variation of parameter, with $y_1 = x, y_2 = x^3$, and $W(y_1,\ y_2) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$.

$$\begin{aligned}
v_1 &= \int \frac{-y_2R(x)}{W(y_1,y_2)}dx = \int \frac{-x^3(2x^2e^x)}{2x^3}dx \\
&= -\int x^2e^x dx = -x^2e^x + 2x6^x - 2e^x \\
v_2 &= \int \frac{y_1R(x)}{W(y_1,y_2)}dx = \int \frac{x(2x^2e^x)}{2x^3} \\
&= \int e^x dx = e^x \\
\therefore y_p &= v_1y_1 + v_2y_2 = (-x^2e^x + 2x6^x - 2e^x)(x) + (e^x)(x^3) = 2x^2e^x - 2xe^x
\end{aligned}$$

The general solution is

$$y = y_g + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$$

Exercise 2.4 Find the general solution of

(a) $y'' - 4y' + 4y = e^{2x}$

(b) $y'' + 4y = \sec 2x; y(0) = 1, y'(0) = 2$

(c) $y'' - 2y' + y = e^x \ln x, x > 0$

(d) $x^2y'' - xy' + y = 2x$

(e) $x^2y'' - 2xy' + 2y = x^4e^x$

(f) $x^2y'' + xy' - y = \ln x$

2.5 System of Differential equation

Definition 2.5.1 A system of DE of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1(t)+a_{12}(t)x_2(t)+\ldots+a_{1n}(t)x_n(t)+f_1(t)\\
\frac{dx_2}{dt} &= a_{21}(t)x_1(t)+a_{22}(t)x_2(t)+\ldots+a_{2n}(t)x_n(t)+f_2(t)\\
&\vdots \qquad\qquad\qquad \vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1(t)+a_{n2}(t)x_2(t)+\ldots+a_{nn}(t)x_n(t)+f_n(t)
\end{aligned} \tag{2.9}$$

where the $a_{ij}(t) and f_i(t)$ are specified functions on an interval I, is called a first-order linear system. If $f_1 = f_2 = \ldots = f_n = 0$, then the system is called homogeneous. Otherwise, it is called nonhomogeneous.

■ **Example 2.10** An example of a nonhomogeneous first-order linear system is

$$\begin{aligned}
\frac{dx_1}{dt} &= e^t x_1 + t^2 x_2 + \sin t\\
\frac{dx_1}{dt} &= t x_1 + 3x_2 - \cos t
\end{aligned}$$

The associated homogeneous system is

$$\begin{aligned}
\frac{dx_1}{dt} &= e^t x_1 + t^2 x_2\\
\frac{dx_1}{dt} &= t x_1 + 3x_2
\end{aligned}$$

■

Definition 2.5.2 By a solution to the system (2.9) on an interval I we mean an ordered n-tuple of functions $x_1(t)$, $x_2(t)$, ..., $x_n(t)$, which, when substituted into the left-hand side of the system, yield the right-hand side for all t in I.

Definition 2.5.3 Solving the system (2.9) subject to n auxiliary conditions imposed at the same value of the independent variable is called an initial-value problem (IVP). Thus, the general form of the auxiliary conditions for an IVP is:

$$x_1(t_0) = \alpha_1, x_2(t_0) = \alpha_2, \ldots, x_n(t_0) = \alpha_n,$$

where α_1, α_2, ..., α_n are constants.

2.5.1 Homogeneous Linear System

consider the homogeneous linear system

$$\begin{aligned}
\frac{dx}{dt} &= a_{11}(t)x(t)+a_{12}(t)y(t)\\
\frac{dy}{dt} &= a_{21}(t)x(t)+a_{22}(t)x(t)
\end{aligned} \tag{2.10}$$

Theorem 2.5.1 If the homogeneous linear system (2.10) has two solution

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \text{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases} \tag{2.11}$$

on $[a,\ b]$, then

$$\begin{cases} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{cases} \tag{2.12}$$

is also a solution of (2.10) on $[a,\ b]$ for arbitrary constants c_1 and c_2 and this solution (2.12) is the general solution of (2.10) if

$$\begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \tag{2.13}$$

dose not vanish on $[a,\ b]$

Example 2.11 Show that

$$\begin{cases} x = -2e^{5t} \\ y = e^{5t} \end{cases}, \quad \text{and} \quad \begin{cases} x = 4e^{-t} \\ y = e^{-t} \end{cases}$$

is a solution to

$$\begin{aligned} x' &= x - 8y \\ y' &= -x + 3y \end{aligned}$$

on $(-\infty,\ \infty)$. Find the general solution of this system and obtain the particular solution for which

$$x(0) = 0, \ \ y(0) = 6$$

2.5.2 Non-homogeneous Linear System

Theorem 2.5.2 If the two solutions (2.11) of the homogeneous system (2.10) are linearly independent on $[a,\ b]$ and if

$$\begin{cases} x = x_p(t) \\ y = y_p(t) \end{cases}$$

is any particular solution of the non-homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x(t) + a_{12}(t)y(t) + f_1(t) \\ \frac{dy}{dt} &= a_{21}(t)x(t) + a_{22}(t)x(t) + f_2(t) \end{aligned} \tag{2.14}$$

on $[a,\ b]$ then

$$\begin{cases} x = c_1x_1(t) + c_2x_2(t) + x_p(t) \\ y = c_1y_1(t) + c_2y_2(t) + y_p(t) \end{cases}$$

is the general solution of (2.14) on $[a,\ b]$

2.6 Operator method for Linear System with constant coefficients

Consider the linear system of

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x(t)+a_{12}y(t)+f_1(t)\\ \frac{dy}{dt} &= a_{21}x(t)+a_{22}x(t)+f_2(t)\end{aligned}$$

This system can be written in the equivalent form

$$(D-a_{11})x \;-\; a_{12}y \;=\; f_1(t) \tag{2.15}$$

$$-a_{21}x \;+\; (D-a_{22})y \;=\; f_2(t) \tag{2.16}$$

where D is the differential operator $\frac{d}{dt}$. The idea behind the solution technique is that we can now easily eliminate y between these two equations by operating on equation (2.15) with $D-a_{22}$, multiplying equation (2.16) by a_{12}, and adding the resulting equations. This yields a second-order constant coefficient linear differential equation for x only. Substituting the expression thereby obtained for x into equation (2.15) will then yield y.

Example 2.12 Solve the IVP

$$\begin{aligned}x' &= x+2y\\ y' &= 2x-2y, \quad x(0)=1,\ y(0)=0\end{aligned}$$

■

Solution: Rewriting the system in operator form as

$$(D-1)x \;-\; 2y \;=\; 0 \tag{2.17}$$

$$-2x \;+\; (D+2)y \;=\; 0 \tag{2.18}$$

To eliminate y between these two equations, we first operate on equation (2.17) with $D+2$ to obtain

$$(D+2)(D-1)x-2(D+2)y=0$$

Adding twice equation (2.18) to this equation eliminates y and yields

$$(D+2)(D-1)x-4x=0 \implies (D^2+D-6)x=0$$

This constant coefficient DE has auxiliary polynomial

$$m^2+m-6=0 \implies (m+3)(m-2)=0 \implies m=-3 \text{ or } m=2$$

Hence, $x=c_1e^{-3t}+c_2e^{2t}$

We now determine y. From equation (2.17), we have

$$y = \frac{1}{2}\left(\frac{d}{dt}(c_1e^{-3t}+c_2e^{2t})-(c_1e^{-3t}+c_2e^{2t})\right)=\frac{1}{2}\left(-4c_1e^{-3t}+c_2e^{2t}\right)$$

Hence, the solution to the given system of DE is

$$\begin{cases}x=c_1e^{-3t}+c_2e^{2t}\\ y=\frac{1}{2}\left(-4c_1e^{-3t}+c_2e^{2t}\right)\end{cases}$$

where c_1 and c_2 are arbitrary constants. Imposing the two initial conditions yields the following equations for determining c_1 and c_2: $\quad c_1+c_2=1,\ \ -4c_1+c_2=0 \implies c_1=\frac{1}{5},\ \ c_2=\frac{4}{5}$

Hence, the particular solution is

$$\begin{cases}x=\frac{1}{5}\left(e^{-3t}+4e^{2t}\right)\\ y=\frac{2}{5}\left(e^{2t}-e^{-3t}\right)\end{cases}$$

Example 2.13 Solve the IVP

$$\begin{aligned} x' + y &= 2\cos t \\ x + y' &= 0 \quad x(0) = 0,\ y(0) = -1 \end{aligned}$$

Solution: Rewriting the system in operator form as

$$Dx + y = 2\cos t \tag{2.19}$$

$$x + Dy = 0 \tag{2.20}$$

To eliminate x between these two equations, we first multiply on equation (2.20) with D and then subtract from (2.19) to obtain

$$D^2y - y = -2\cos t \implies \frac{d^2y}{dt^2} - y = -2\cos t$$

which is a non-homogeneous second order ode.
The general solution of the corresponding homogeneous equation is :

$$y_g(t) = c_1e^t + c^2e^{-t}$$

and the fixed particular solution is

$$y_p(t) = \cos t.$$

Hence, the general solution is

$$y(t) = c_1e^t + c_2e^{-t} + \cos t$$

Differentiating, we get

$$y'(t) = c_1e^t - c_2e^{-t} - \sin t$$

From equation (2.20) we obtain

$$x(t) = -c_1e^t + c_2e^{-t} + \sin t.$$

Therefore, the general solution of the given system of equation is

$$\begin{cases} x = -c_1e^t + c_2e^{-t} + \sin t \\ y = c_1e^t + c_2e^{-t} + \cos t \end{cases}$$

where c_1 and c_2 are arbitrary constants. Imposing the two initial conditions yields the following equations for determining c_1 and c_2:

$$-c_1 + c_2 = 0, \text{ and } c_1 + c_2 + 1 = -1 \quad \implies \quad c_1 = -1, \ \ c_2 = 1$$

Hence, the particular solution is

$$\begin{cases} x = e^t + e^{-t} + \sin t \\ y = -e^t + e^{-t} + \cos t \end{cases}$$

Exercise 2.5 Solve

(a)
$$\begin{aligned} \frac{dx}{dt} + 4x + 3y &= t \\ \frac{dy}{dt} + 2x + 5y &= e^t \end{aligned}$$

(b)
$$\begin{aligned} \frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2 \end{aligned}$$

(c)
$$\begin{aligned} Dx - 3y &= 6a\sin t \\ 3x + Dy &= 0 \end{aligned}$$
subject to $x(0) = a,\ y(0) = 0$

2.7 Applications of Second-Order Differential Equations

2.7.1 Spring/Mass System

HOOKE'S LAW: states that the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s. i.e.,

$$F = ks$$

where k is a constant of proportionality called the **spring constant**.

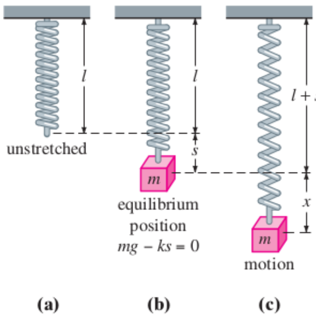


Figure 2.1: Spring/mass system

NEWTON'S SECOND LAW After a mass m is attached to a spring, it stretches the spring by an amount s and attains a position of equilibrium at which its weight $W = mg$ is balanced by the restoring force ks. If the mass is displaced by an amount x from its equilibrium position, the restoring force of the spring is then $k(x+s)$.
Assuming that there are no retarding forces acting on the system and assuming that the mass vibrates free of other external forces '**free motion**' we can equate Newton's second law with the net, or resultant, force of the restoring force and the weight:

$$m\frac{d^2x}{dt^2} = -k(x+s)+mg = -kx+mg-ks = -kx \tag{2.21}$$

The negative sign in (2.21) indicates that the restoring force of the spring acts opposite to the direction of motion. Furthermore, we adopt the convention that displacements measured below the equilibrium position are positive. See Figure 2.2

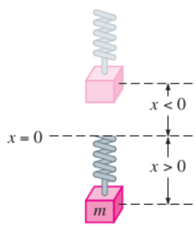


Figure 2.2: Direction below the equilibrium position is positive.

Differential equation of free undamped motion : By dividing (2.21) by the mass m, we obtain the second-order differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2.22}$$

where $\omega^2 = \frac{k}{m}$. Equation (2.22) is said to describe simple harmonic motion or free undamped motion. If the system starts at $t = 0$ with an initial position x_0 and initial velocity x_1, we have initial condition's $x(0) = x_0$, and $x'(0) = x_1$.
Thus, the general solution of (2.22) is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{2.23}$$

The natural frequency is $f = \frac{\omega}{2\pi}$ and the **period** of motion is $T = \frac{1}{f} = \frac{2\pi}{\omega}$. The number $\omega = \sqrt{\frac{k}{m}}$ (measured in radians per second) is called the circular frequency of the system.
Equation (2.23) can be re-expressed as

$$x(t) = A\cos(\omega t - \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$ is **Amplitude** and
$\phi = \tan^{-1}(c_2/c_1)$, is **phase angle**. Where $\sin\phi = \frac{c_2}{A}$, $\cos\phi = \frac{c_2}{A}$
Differential equation of free damped motion: In the study of mechanics, damping forces acting on a body are considered to be proportional to a power of the instantaneous velocity. In particular, we shall assume throughout the subsequent discussion that this force is given by a constant multiple of $\frac{dx}{dt}$. When no other external forces are impressed on the system, it follows from Newton's second law that

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \tag{2.24}$$

where β is a positive damping constant and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion.
Dividing (2.24) by the mass m, we find that the differential equation of free damped motion is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \tag{2.25}$$

where $2\lambda = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$.
Case I: If $\lambda^2 - \omega^2 > 0$. The system is said to be **overdamped** because the damping coefficient β is large when compared to the spring constant k.
The corresponding solution of (2.25) is

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right)$$

This equation represents a smooth and nonoscillatory motion.
Case II: If $\lambda^2 - \omega^2 = 0$. The system is said to be **critically damped** because any slight decrease in the damping force would result in oscillatory motion. The general solution of (2.25) is

$$x(t) = e^{-\lambda t}(c_1 + c_2 t) \tag{2.26}$$

The motion is quite similar to that of an overdamped system. It is also apparent from (2.26) that the mass can pass through the equilibrium position at most one time.
Case III: If $\lambda^2 - \omega^2 < 0$. The system is said to be **underdamped**, since the damping coefficient is small in comparison to the spring constant. Thus the general solution of equation (2.25) is

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\lambda^2 - \omega^2}t + c_2 \sin\sqrt{\lambda^2 - \omega^2}t)$$

The motion is oscillatory; but because of the coefficient $e^{-\lambda t}$ the amplitudes of vibration $\to 0$ as $t \to \infty$

Example 2.14 A spring with a mass of 2 *kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. If the spring is stretched to a length of 0.7 *m* and then released with initial velocity 0, find the position of the mass at any time t. ■

Solution: From Hooke's Law, the force required to stretch the spring is

$$F = ks \implies k(0.2) = 25.6 \implies k = 128$$

Using this value of the spring constant k, together with $m = 2$ in Equation (2.21), we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

The solution of this equation is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

We are given the initial condition that $x(0) = 0.2$. Hence, $x(0) = c_1$. Therefore, $c_1 = 0.2$.

Differentiating the solution, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Since the initial velocity is given as, $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = 0.2 \cos 8t$$

Example 2.15 Suppose that the spring of Example 2.14 is immersed in a fluid with damping constant $\beta = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s. ■

Solution: From Example 2.14 the mass is $m = 2$ and the spring constant is $k = 128$, so the differential equation (2.24) becomes

$$\begin{aligned} 2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x &= 0 \\ \frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x &= 0 \end{aligned}$$

The auxiliary equation is $r^2 + 20r + 64 = 0 \implies (r+4)(r+16) = 0$ with roots -4 and -16 , so the motion is overdamped and the solution is

$$x(t) = c_1 e^{-4t} + c_2 e^{-16t}$$

We are given that $x(0) = 0$, so $c_1 + c_2 = 0$. Differentiating, we get

$$x'(t) = -4c_1 e^{-4t} - 16c_2 e^{-16t} \implies x'(0) = -4c_1 - 16c_2 = 0.6$$

Hence, we get $c_1 = c_2 = 0.05$

Therefore,

$$x(t) = 0.05(e^{-4t} + e^{-16t})$$

Exercise 2.6

1. A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of $3\sqrt{2}$ ft/s. **(Take** $g = 32$ ft/s^2**)**
2. A mass of 1 *kg*, when attached to a spring, stretches it 2 *m* and then comes to rest in the equilibrium position. Starting at $t = 0$, an external force equal to $f(t) = 8\sin 4t$ is applied to the system. Find the equation of motion if the surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity.

■

2.7.2 Electric Circuit

Consider the RLC Circuit below

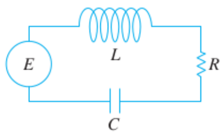


Figure 2.3: LRC series circuit.

Kirchhoff's Law The algebric sum of the voltage drops in a simple closed circuit is zero. The voltage drop across the resistor, capacitor and inductor are given RI, $\frac{1}{c}q$, and $L\frac{dI}{dt}$ respectively. Hence

$$RI + L\frac{dI}{dt} + \frac{1}{c}q = E(t) \tag{2.27}$$

Since $I = \frac{dq}{dt} \Rightarrow \frac{dI}{dt} = \frac{d^2q}{dt^2}$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{cL}q = \frac{E(t)}{L} \tag{2.28}$$

The initial conditions may be $q(0) = q_0$, $\frac{dq}{dt}|_{t=0} = I(0) = I_0$

To obtain a differential equation for current differentiating equ (2.27) with respect to time t,

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{c}\frac{dq}{dt} = \frac{dE(t)}{dt}$$

Since $\frac{dq}{dt} = I$

$$\Rightarrow \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{cL}I = \frac{E(t)}{dt} \tag{2.29}$$

The initial conditions may be $I(0) = I_0$, and $\frac{dI}{dt}|_{t=0} = \frac{1}{L}$.

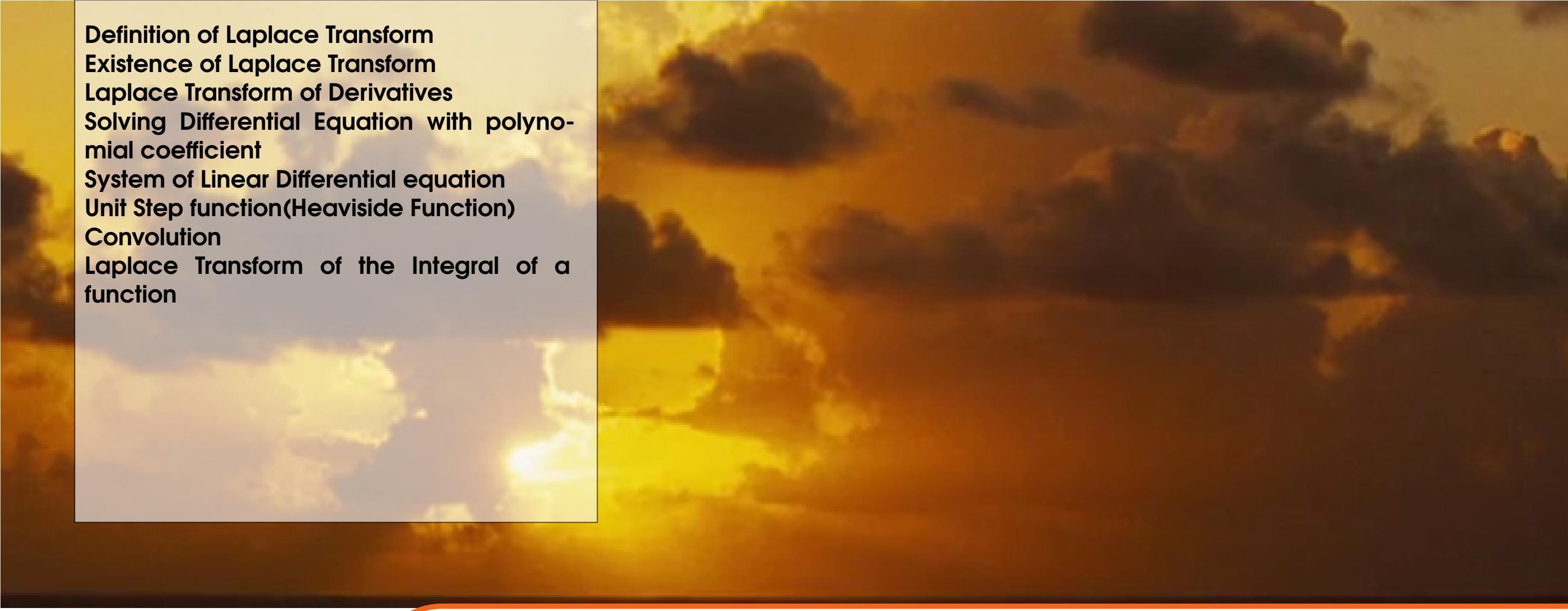
If $E(t) = 0$, the electrical vibrations of the circuit are said to be free.

We say that the circuit is

$$\textbf{overdamped} \text{ if } R^2 - \frac{4L}{C} > 0$$
$$\textbf{critically damped} \text{ if } R^2 - \frac{4L}{C} = 0$$
$$\textbf{underdamped} \text{ if } R^2 - \frac{4L}{C} < 0$$

Example 2.16 Find the charge $q(t)$ on the capacitor in an LRC series circuit when $L = 0.25\ H$, $R = 10 ohms$, $C = 0.001$ farad , $E(t) = 0$, $q(0) = q_0$ coulombs, and $I(0) = 0$.

Example 2.17 Find the charge and current at time t in an LRC series circuit when $L = 1\ H$, $R = 40\omega$, $C = 16\times 10^{-4} = 0.0016\ F$, $E(t) = 100\cos 10t$, and the initial charge and current are both 0.



Definition of Laplace Transform
Existence of Laplace Transform
Laplace Transform of Derivatives
Solving Differential Equation with polynomial coefficient
System of Linear Differential equation
Unit Step function(Heaviside Function)
Convolution
Laplace Transform of the Integral of a function

3 — Laplace Transform

3.1 Definition of Laplace Transform

Definition 3.1.1 The Laplace transform of a function $f(t)$, denoted by $F(s) = \mathscr{L}(f(t))$ is a function defined by

$$F(s) = \mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt \tag{3.1}$$

for all s such that this integral converges.

Definition 3.1.2 The orginal function $f(t)$ in (3.1) is called the inverse transform or invers of $F(s)$ and will be denoted by $\mathscr{L}^{-1}(F(s))$

$$\text{i.e., } f(t) = \mathscr{L}^{-1}(F(s))$$

Example 3.1 Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$

Solution: From (3.1) we obtain

$$\begin{aligned}
\mathscr{L}(f(t)) &= \mathscr{L}(1) = \int_0^\infty e^{-st}dt = \lim_{n\to\infty}\int_0^n e^{-st}dt = \lim_{n\to\infty}\left(\frac{-1}{s}e^{-st}\right)\Bigg|_0^n \\
&= \lim_{n\to\infty}\left(\frac{-1}{s}(e^{-sn}-1)\right) \\
&= \frac{1}{s}, \ (s>0)
\end{aligned}$$

Example 3.2 Let $f(t) = e^{at}$ when $t \geq 0$, where a is constant. Find $\mathscr{L}(f)$

Solution: By definition

$$\begin{aligned}
F(s) = \mathscr{L}(f(t)) &= \mathscr{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}dt = \lim_{n\to\infty}\int_0^n e^{-(s-a)t}dt = \lim_{n\to\infty}\left(\frac{-1}{s-a}e^{-(s-a)t}\right)\Bigg|_0^n \\
&= \frac{1}{s-a}, \ (s>a)
\end{aligned}$$

Theorem 3.1.1 — Linearity of the Laplace Transform The Laplace transform is linear operation; that is, for any function $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a & b,

$$\mathscr{L}\{af(t)+bg(t)\} = a\mathscr{L}(f(t))+b\mathscr{L}(g(t))$$

Example 3.3 Let $f(t) = \sin\omega t$. Find $\mathscr{L}(f(t))$

Solution: Since $\mathscr{L}(e^{at}) = \dfrac{1}{s-a}$, set $a = i\omega$ with $i = \sqrt{-1}$

$$\Rightarrow \quad \mathscr{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

Since $e^{i\omega t} = \cos\omega t + i\sin\omega t$ (Euler's Formule) and by theorem 3.1.1, we obtain

$$\begin{aligned} \mathscr{L}(e^{i\omega t}) &= \mathscr{L}(\cos\omega t + i\sin\omega t) \\ &= \mathscr{L}(\cos\omega t) + i\mathscr{L}(\sin\omega t) \end{aligned}$$

Equating the real and imaginary parts of these two equations, we get

$$\mathscr{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \text{ and } \mathscr{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

	$f(t)$	$\mathscr{L}(f(t)) = F(s)$	Domain
1	c (constant)	$\frac{c}{s}$	$s > 0$
2	t	$\frac{1}{s^2}$	$s > 0$
3	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
4	e^{kt}	$\frac{1}{s-k}$	$s > k$
5	$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$
6	$\cos kt$	$\frac{s}{s^2+k^2}$	$s > 0$
7	$\cosh kt$	$\frac{s}{s^2-k^2}$	$s > k$
8	$\sinh kt$	$\frac{k}{s^2-k^2}$	$s > k$

Theorem 3.1.2 — (First Shifting Theorem) If $f(t)$ has the transform $F(s)$ (where $s > k$), then $e^{at}f(t)$ has the transform $F(s-a)$ (where $s-a > k$)

$$\begin{aligned} \text{i.e., } \mathscr{L}(e^{at}f(t)) &= F(s-a) \text{ or} \\ e^{at}f(t) &= \mathscr{L}^{-1}(F(s-a)) \end{aligned}$$

Example 3.4 Compute $\mathscr{L}(e^{2t}\cos 3t)$

Solution: Since $F(s) = \mathscr{L}(\cos 3t) = \dfrac{s}{s^2+9}$ and $a = 2$, we have $\mathscr{L}(e^{2t}\cos 3t) = \dfrac{s-2}{(s-2)^2+9}$

Example 3.5 Find $\mathscr{L}^{-1}\left(\dfrac{1}{s-4} - \dfrac{6}{(s-4)^2}\right)$

Solution: Since $\mathscr{L}^{-1}\left(e^{4t}\right) = \dfrac{1}{s-4}$ and $\mathscr{L}(t) = \dfrac{1}{s^2} \Rightarrow \mathscr{L}(te^{4t}) = \dfrac{1}{(s-4)^2}$

$$\therefore \quad \mathscr{L}^{-1}\left(\frac{1}{s-4} - \frac{6}{(s-4)^2}\right) = \mathscr{L}^{-1}\left(\frac{1}{s-4}\right) - 6\mathscr{L}^{-1}\left(\frac{1}{(s-4)^2}\right) = e^{4t} - 6te^{4t}$$

Exercise 3.1 Compute

1. $\mathscr{L}(e^{at}t^n)$
2. $\mathscr{L}^{-1}\left(\dfrac{1}{s^2+4s+13}\right)$
3. $\mathscr{L}^{-1}\left(\dfrac{2s+4}{s^2+4s+5}\right)$

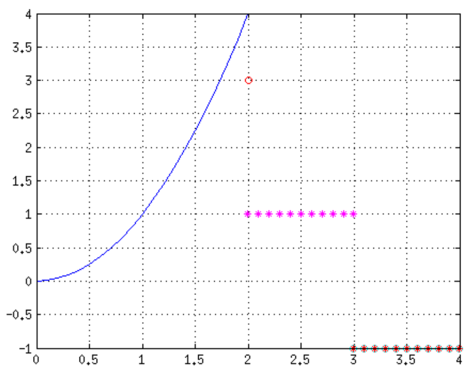
3.2 Existence of Laplace Transform

Definition 3.2.1 A function f has a jump discontinuous at a point t_0 if the function has different (finite) limits at t approaches t_0 from the left and from the right or if the two limits are equal but different from $f(t_0)$. Note that $f(t_0)$ may or may not be equal to either

$$\lim_{t \to t_0^+} f(t) \text{ or } \lim_{t \to t_0^-} f(t)$$

Definition 3.2.2 A function f defined on $(0, \infty)$ is picewise continuous if it is continuous on every finite interval $0 \leq t \leq \infty$, except possibly at finitely many points where it has jump discontinuties

Example 3.6 Let $\begin{cases} t^2 & \text{for} \quad 0 \leq t < 2 \\ 3 & \text{for} \quad t = 2 \\ 1 & \text{for} \quad 2 < t \leq 2 \\ -1 & \text{for} \quad 3 < t \leq 4 \end{cases}$



Theorem 3.2.1 — Existence Theorem Let $f(t)$ be a function which is picewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{kt}, \quad \forall t \geq 0 \tag{3.2}$$

and for some constant k and M. Then the Laplace transform of $f(t)$ exists for all $s > k$

Proof. Since f is piecewise continuous, $e^{-st}f(t)$ has a finite integral over any finite interval on $t \geq 0$, and

$$\begin{aligned} |\mathscr{L}(f(t))| = \left|\int_0^\infty e^{-st}f(t)dt\right| &\leq \int_0^\infty e^{-st}|f(t)|dt \\ &\leq \int_0^\infty Me^{-st}e^{kt}dt = M\int_0^\infty e^{-(s-k)t}dt \\ &= \frac{M}{s-k}, \quad s > k \end{aligned}$$

$\mathscr{L}(f(t))$ exists. (comparison theorem) ■

3.3 Laplace Transform of Derivatives

Theorem 3.3.1 Suppose that $f(t)$ is contiuous for all $t \geq 0$, satisfies the condition

$$|f(t)| \leq Me^{kt}$$

for some k and M, and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative $f'(t)$ exists when $s > k$ and

$$\mathscr{L}(f'(t)) = s\mathscr{L}(f(t)) - f(0) \tag{3.3}$$

Proof. Suppose $f'(t)$ is continuous for all $t \geq 0$. Integrating by parts

$$\begin{aligned} \mathscr{L}(f'(t)) &= \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)|_0^\infty + \int_0^\infty se^{-st}f(t)dt \\ &= 0 - f(0) + s\int_0^\infty e^{-st}f(t)dt = -f(0) + s\mathscr{L}(f(t)) \\ \therefore\ \mathscr{L}(f'(t)) &= s\mathscr{L}(f(t)) - f(0) \end{aligned}$$

■

Theorem 3.3.2 — Laplace transform of the derivative of any order n Let $f(t)$ and its derivatives $f'(t),\ f''(t),\ \ldots,\ f^{(n-1)}(t)$ be continuous functions for all $t \geq 0$, satisfies the condition

$$|f(t)| \leq Me^{kt}$$

for some k and M, and let the derivative $f^{(n)}(t)$ be piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative $f^{(n)}(t)$ exists when $s > k$ and is given by

$$\mathscr{L}(f^{(n)}(t)) = s^n\mathscr{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots, -f^{(n-1)(0)} \tag{3.4}$$

For $n = 2$, $\mathscr{L}(f''(t)) = s^2\mathscr{L}(f(t)) - sf(0) - f'(0)$

■ **Example 3.7** Let $f(t) = \cos^2 t$. Find $\mathscr{L}(f(t))$ ■

Solution: We have $f(0) = 1,\ f'(t) = -2\cos t \sin t = -\sin 2t$

$$\begin{aligned}
\mathscr{L}(f'(t)) &= \mathscr{L}(-\sin 2t) = s\mathscr{L}(f(t)) - f(0) \\
\Rightarrow\ \frac{-2}{s^2+4} &= s\mathscr{L}(\cos^2 t) - 1 \ \Rightarrow\ s\mathscr{L}(\cos^2 t) = 1 - \frac{2}{s^2+4} \\
\therefore\ \mathscr{L}(\cos^2 t) &= \frac{s^2+2}{s(s^2+4)}
\end{aligned}$$

■ **Example 3.8** Let $f(t) = t\sin\omega t$. Find $\mathscr{L}(f(t))$ ■

Solution: We have $f(0) = 0,\ f'(t) = \sin\omega t + t\omega\cos\omega t \Rightarrow f'(0) = 0$
$f''(t) = 2\omega\cos\omega t - t\omega^2\sin\omega t$

$$\begin{aligned}
&\mathscr{L}(f''(t)) = s^2\mathscr{L}(f(t)) - sf(0) - f'(0) \\
\Rightarrow\ &\mathscr{L}(2\omega\cos\omega t - t\omega^2\sin\omega t) = s^2\mathscr{L}(t\sin\omega t) - s.0 - 0 \\
\Rightarrow\ &2\omega\mathscr{L}(\cos\omega t) - \omega^2\mathscr{L}(t\sin\omega t) = s^2\mathscr{L}(t\sin\omega t) \\
\Rightarrow\ &\frac{2\omega s}{s^2+\omega^2} = (s^2+\omega^2)\mathscr{L}(t\sin\omega t) \\
\Rightarrow\ &\mathscr{L}(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}
\end{aligned}$$

■ **Example 3.9** Solve the IVP
a) $y'' - 4y = 0,\ \ y(0) = 1,\ y'(0) = 2$ b) $y'' - 3y' + 2y = 4t - 6,\ \ y(0) = 1,\ y'(0) = 3$ ■

Solution: a) Taking Laplace transform both sides and using differentiation property, we have

$$\begin{aligned}
\mathscr{L}(y'' - 4y) &= \mathscr{L}(y'') - 4\mathscr{L}(y) = 0 \ \Rightarrow\ s^2\mathscr{L}(y) - sy(0) - y'(0) - 4\mathscr{L}(y) = 0 \\
&\Rightarrow\ (s^2-4)\mathscr{L}(y) - s - 2 = 0 \\
&\Rightarrow\ \mathscr{L}(y) = \frac{s+2}{s^2-4} = \frac{1}{s-2} \\
&\Rightarrow\ y = \mathscr{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}
\end{aligned}$$

$\therefore\ y(t) = e^{2t}$

b) $y'' - 3y' + 2y = 4t - 6,\ \ y(0) = 1,\ y'(0) = 3$ Taking Laplace transform both sides

$$\begin{aligned}
&\mathscr{L}(y'' - 3y' + 2y) = \mathscr{L}(4t - 6) \\
\Rightarrow\ &\mathscr{L}(y'') - 3\mathscr{L}(y') + 2\mathscr{L}(y) = 4\mathscr{L}(t) - \mathscr{L}(6) \\
\Rightarrow\ &s^2\mathscr{L}(y) - sy(0) - y'(0) - 3s\mathscr{L}(y) + 3y(0) + 2\mathscr{L}(y) = \frac{4}{s^2} - \frac{6}{s}
\end{aligned}$$

$$\Rightarrow \quad (s^2-3s+2)\mathscr{L}(y)-s=\frac{4-6s}{s^2}$$
$$\Rightarrow \quad (s-2)(s-1)\mathscr{L}(y)=\frac{4-6s}{s^2}+s=\frac{4-6s+s^3}{s^2}$$
$$\Rightarrow \quad \mathscr{L}(y)=\frac{(s-2)(s^2+2s-2)}{s^2(s-2)(s-1)}=\frac{s^2+2s-2}{s^2(s-1)}$$
$$\Rightarrow \quad \mathscr{L}(y)=\frac{s^2}{s^2(s-1)}+\frac{2(s-1)}{s^2(s-1)}=\frac{1}{s-1}+\frac{2}{s^2}$$
$$\Rightarrow \quad y(t)=\mathscr{L}^{-1}\left(\frac{1}{s-1}\right)+2\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)$$

$\therefore \; y(t)=e^t+2t$

Exercise 3.2 Solve the IVP
1. $y''+4y=0, \;\; y(0)=1, \; y'(0)=2$
2. $y''-5y'+6y=e^{-t}, \;\; y(0)=0, \; y'(0)=2$
3. $y'+4y=\cos t, \;\; y(0)=0$
4. $y''+4y'+3y=e^t, \;\; y(0)=1, \; y'(0)=2$

3.4 Solving Differential Equation with polynomial coefficient

Theorem 3.4.1 Let $\mathscr{L}(f(t))=F(s)$ for $s>a$, and suppose that F is differentiable. Then

$$\mathscr{L}(tf(t))=-\frac{d}{ds}\mathscr{L}(f(t)) \tag{3.5}$$

Proof. $F(s)=\int_0^\infty e^{-st}f(t)dt$

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{d}{ds}e^{-st}f(t)dt=\int_0^\infty -te^{-st}f(t)dt=-\mathscr{L}(tf(t))$$

■

Ingeneral,

$$\mathscr{L}(t^nf(t))=(-1)^n\frac{d^n}{ds^n}F(s) \tag{3.6}$$

Example 3.10 Find $\mathscr{L}(e^{-t}t\sin 2t)$

Solution:

$$\begin{aligned}\mathscr{L}(e^{-t}t\sin 2t) &= -\frac{d}{ds}\mathscr{L}(e^{-t}\sin 2t)\\ &= -\frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right)\\ &= \frac{4(s+1)}{((s+1)^2+4)^2}\end{aligned}$$

Example 3.11 Solve the equation with variable coefficients

$$ty'' - ty' - y = 0, \;\; y(0) = 0, \; y'(0) = 3$$

Solution: Let $Y(s) = \mathscr{L}(y(t))$

$$\begin{aligned}
\mathscr{L}(ty''(t)) &= -\frac{d}{ds}\mathscr{L}(y'') = -\frac{d}{ds}\left(s^2\mathscr{L}(y) - sy(0) - y'(0)\right) \\
&= -\frac{d}{ds}\mathscr{L}\left(s^2Y(s) - s.0 - 3\right) \\
&= -(2sY(s) + s^2Y'(s))
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}(ty'(t)) &= -\frac{d}{ds}\mathscr{L}(y') = -\frac{d}{ds}(s\mathscr{L}(y) - y(0)) \\
&= -\frac{d}{ds}(sY(s)) = -(Y(s) + sY'(s))
\end{aligned}$$

Thus, $ty'' - ty' - y = 0, \;\; y(0) = 0$. Taking both sides Laplace transform.

$$\begin{aligned}
&\Rightarrow \;\; \mathscr{L}(ty'' - ty' - y) = 0 \;\Rightarrow\; \mathscr{L}(ty'') - \mathscr{L}(ty) - \mathscr{L}(y) = 0 \\
&\Rightarrow \;\; -(Y(s) + sY'(s)) + Y(s) + sY'(s) - Y(s) = 0 \\
&\Rightarrow \;\; (s - s^2)Y'(s) - 2sY(s) = 0 \;\Rightarrow\; s(s-1)Y' = 2sY \\
&\Rightarrow \;\; Y' = \frac{2}{1-s}Y \;\;\Rightarrow\;\; \frac{dY}{ds} = \frac{2}{1-s}Y \\
&\Rightarrow \;\; \frac{dY}{Y} = \frac{2}{1-s}ds \\
&\Rightarrow \;\; \ln Y = -2\ln(s-1) + \ln c \\
&\Rightarrow \;\; Y = \frac{c}{(s-1)^2} \;\;\Rightarrow\;\; \mathscr{L}(y) = \frac{c}{(s-1)^2} \\
&\Rightarrow \;\; y(t) = \mathscr{L}^{-1}\left(\frac{c}{(s-1)^2}\right) = cte^t
\end{aligned}$$

$y(0) = 0$ To find c, $y'(t) = ce^t + cte^t \Rightarrow y'(0) = c = 3$

$$\therefore \;\; y(t) = 3te^t$$

Exercise 3.3 Find

1. $\mathscr{L}(te^{-t}\sin 4t)$
2. $\mathscr{L}(t^2e^{3t}\cos 2t)$

Exercise 3.4 Solve the IVP

1. $ty'' - y' = 2t^2, \;\; y(0) = 0$
2. $ty'' + (4t-2)y' - 4y = 0, \;\; y(0) = 1$
3. $2y'' + ty' - 2y = 10, \;\; y(0) = y'(0) = 0$

3.5 System of Linear Differential equation

Consider

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x(t) + a_{12}(t)y(t) + f(t) \\ \frac{dy}{dt} &= a_{21}(t)x(t) + a_{22}(t)x(t) + g(t) \end{aligned} \tag{3.7}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$
By taking Laplace transform both equations in system (3.7) we can find the solutions of the system.

Example 3.12 Consider the system of initial value equation

$$\begin{aligned} x' + y &= e^{2t} \\ x + y' &= 0 \end{aligned}$$

s.t $x(0) = 0,\ y(0) = 0$

Exercise 3.5 Solve system of differential equation

$$a. \quad \begin{aligned} x' + y &= 2\cos t \\ x + y' &= 0 \end{aligned}$$

s.t $x(0) = 0,\ y(0) = 1$

$$b. \quad \begin{aligned} y_1'' &= y_1 + 3y_2 \\ y_2'' &= 4y_1 - 4e^t \end{aligned}$$

s.t $y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = 1, y_2(0) = 2$

$$c \quad \begin{aligned} y_1' &= -y_2 \\ y_2' &= y_1,\ y_1(0) = 1,\ y_2(0) = 0 \end{aligned}$$

3.6 Unit Step function(Heaviside Function)

A unit step function is defined by

$$U(t-a) = U_a(t) = \begin{cases} 0 & \text{if} \quad t < a \\ & \\ 1 & \text{if} \quad t \geq a \end{cases} \qquad a \geq 0$$

A general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t) & \text{if} \quad 0 \leq t < a \\ h(t) & \text{if} \quad t \geq a \end{cases}$$

is the same as

$$f(t) = g(t) - g(t)U(t-a) + h(t)U(t-a).$$

Similarly, a function of the type

$$f(t) = \begin{cases} 0 \ \text{ if } & 0 \le t < a \\ g(t), \ \text{ if } & a \le t < b \\ 0 \ \text{ if } & t \ge b \end{cases}$$

can be written

$$f(t) = g(t)\left(U(t-a) - U(t-b)\right).$$

The transform of $U(t-a)$ is

$$\begin{aligned} \mathscr{L}(U(t-a)) &= \int_0^\infty e^{-st}U(t-a)dt \\ &= \int_a^\infty e^{-st}.1dt = -\frac{e^{-st}}{s}\Big|_a^\infty \\ \mathscr{L}(U(t-a)) &= \frac{e^{-as}}{s}, \ s > 0 \end{aligned}$$

Theorem 3.6.1 — Second shifting theorem If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathscr{L}(U_a(t)f(t-a)) = e^{-as}F(s) \tag{3.8}$$

Proof.

$$\begin{aligned} \mathscr{L}(U_a(t)f(t-a)) &= \int_0^\infty e^{-st}U_a(t)f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt \qquad \text{Let } \xi = t-a \\ &= \int_a^\infty e^{-s(\xi+a)}f(\xi)d\xi = e^{-sa}\int_a^\infty e^{-s(\xi)}f(\xi)d\xi \\ &= e^{-as}F(s) \end{aligned}$$

■

Example 3.13 Let $f(t) = \begin{cases} 0, & t < 2 \\ t-2, & t \ge 2 \end{cases}$ Find $\mathscr{L}(f(t))$

Solution: $U_2(t) = \begin{cases} 0, & t < 2 \\ 1, & t > 2 \end{cases} \quad \Rightarrow \quad f(t) = U_2(t)(t-2)$

$$\mathscr{L}(U_2(t)(t-2)) = e^{-2s}\mathscr{L}(t) = \frac{e^{-2s}}{s^2}$$

Example 3.14 Find the inverse transform of $F(s) = \dfrac{1+e^{-2s}}{s^2}$

Solution:

$$\begin{aligned} \mathscr{L}^{-1}(F(s)) &= \mathscr{L}^{-1}\left(\frac{1}{s^2} + \frac{e^{-2s}}{s^2}\right) = t + U_2(t)(t-2) \\ &= \begin{cases} t & 0 \le t < 2 \\ 2(t-1) & t \ge 2 \end{cases} \end{aligned}$$

■ **Example 3.15** Solve the IVP

$$y''+y=g(t),\ \ y(0)=0,\ y'(0)=1$$

where $g(t)=\begin{cases}0 & 0\le t<1\\ 1 & 1\le t<2\end{cases}$ ■

Solution: We can express $g(t)$ as $U_1(t)-U_2(t)$.
The Laplace transform of the IVP is

$$\begin{aligned}
& \mathscr{L}(y'')+\mathscr{L}(y)=\mathscr{L}(g(t))\\
\Rightarrow\quad & s^2\mathscr{L}(y)-sy(0)-y'(0)+\mathscr{L}(y)=\mathscr{L}(U_1(t))-\mathscr{L}(U_2(t))\\
\Rightarrow\quad & (s^2+1)\mathscr{L}(y(t))-1=\frac{e^{-s}}{s}-\frac{e^{-2s}}{s}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & \mathscr{L}(y)=\frac{1}{s^2+1}+\frac{e^{-s}}{s(s^2+1)}-\frac{e^{-2s}}{s(s^2+1)}\\
\Rightarrow\quad & y(t)=\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)+\mathscr{L}^{-1}\left(\frac{e^{-s}}{s(s^2+1)}\right)-\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+1)}\right)\\
\Rightarrow\quad & y(t)=\sin t+U_1(t)\left[1-\cos(t-1)\right]-U_2(t)\left[1-\cos(t-2)\right]
\end{aligned}$$

3.7 Convolution

Definition 3.7.1 The convolution of the function f and g written by $f*g$ is defined by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau,\quad \forall t\ge 0 \tag{3.9}$$

Theorem 3.7.1 — The convolution Theorem If $F(s)$ and $G(s)$ are the Laplace transform of $f(t)$ and $g(t)$ respectively, then

$$\mathscr{L}(f*g)(t)=\mathscr{L}(f)\mathscr{L}(g) \tag{3.10}$$

Properties of convolution

1. $f*g=g*f$
2. $f*(g*h)=(f*g)*h$ (associative)
3. $f*(g+h)=(f*g)+(f*h)$ (Distributive)

■ **Example 3.16** Let $H(s)=\dfrac{1}{(s^2+\omega^2)^2}$. Find $h(t)$ ■

Solution: We have $\mathscr{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin\omega t}{\omega}$.

$$\begin{aligned}
h(t) &= \frac{\sin\omega t}{\omega} * \frac{\sin\omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin\omega\tau \sin\omega(t-\tau)d\tau \\
&= \frac{1}{\omega^2}\int_0^t \sin\omega\tau \sin\omega(t-\tau)d\tau \\
&= \frac{1}{\omega^2}\int_0^t \sin\omega\tau[\sin\omega t\cos\omega\tau - \sin\omega\tau\cos\omega t]d\tau \\
&= \frac{\sin\omega t}{\omega^2}\int_0^t \sin\omega\tau\cos\omega\tau - \frac{\cos\omega t}{\omega^2}\int_0^t \sin^2\omega\tau d\tau \\
&= \frac{\sin\omega t}{\omega^2}\int_0^t \sin\omega\tau\cos\omega\tau - \frac{\cos\omega t}{\omega^2}\int_0^t \left(\frac{1}{2}+\frac{\cos 2\omega\tau}{2}\right)d\tau \\
&= \frac{\sin\omega t}{\omega^2}\left[\frac{\sin^2\omega\tau}{2\omega}\right]_0^\tau - \frac{\cos\omega t}{\omega^2}\left[\frac{1}{2}t - \frac{\sin 2\omega\tau}{4\omega}\right]_0^\tau \\
&= \frac{\sin\omega t}{\omega^2}\left[\frac{\sin^2\omega t}{2\omega}\right] - \frac{t\cos\omega t}{2\omega^2} + \frac{\cos\omega t\sin 2\omega t}{4\omega^3} \\
&= \frac{\sin\omega t}{\omega^2}\left[\frac{\sin^2\omega t}{2\omega}\right] - \frac{t\cos\omega t}{2\omega^2} + \frac{\cos^2\omega t\sin\omega t}{2\omega^3} \\
&= \frac{\sin\omega t}{2\omega^3}\left[\sin^2\omega t + \cos^2\omega t\right] - \frac{t\cos\omega t}{2\omega^2} \\
&= \frac{1}{2\omega^2}\left[\frac{\sin\omega t}{\omega} - t\cos\omega t\right]
\end{aligned}$$

■ **Example 3.17** Solve the initial value problem

$$y'' + 4y' + 13y = 2e^{-2t}\sin 3t, \quad y(0) = 1, \; y'(0) = 0$$

■

Solution:

Integral equation

An equation of the form

$$y(t) = f(t) + \lambda\int_0^t K(t,\tau)y(\tau)d\tau \tag{3.11}$$

is called a Volterra integral equation, where λ is a parameter and $K(t,\tau)$ is called the kernel of the integral equation. The Laplace transform is well suited to the solution of such integral equations when the kernel $K(t,\tau)$ has a special form that depends on t and τ only through the difference $t-\tau$, because then $K(t,\tau) = K(t-\tau)$ and the integral in (3.11) becomes a convolution integral.

■ **Example 3.18** Solve the Volterra integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau$$

■

Solution: Taking laplace tarnsform both sides, we get

$$\begin{aligned}
\mathscr{L}(y(t)) &= \mathscr{L}\left(2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau\right) \\
&= \frac{2}{1+s} + \mathscr{L}\left(\int_0^t \sin(t-\tau)y(\tau)d\tau\right) \\
&= \frac{2}{1+s} + \frac{\mathscr{L}(y(t))}{s^2+1} \\
\left(1 - \frac{1}{s^2+1}\right)\mathscr{L}(y(t)) &= \frac{2}{1+s} \quad \Rightarrow \quad \left(\frac{s^2}{s^2+1}\right)\mathscr{L}(y(t)) = \frac{2}{1+s} \\
\mathscr{L}(y(t)) &= \frac{2(s^2+1)}{s^2(s+1)} = \frac{2}{s^2} - \frac{2}{s} + \frac{4}{1+s} \\
y(t) &= \mathscr{L}^{-1}(\frac{2}{s^2}) - \mathscr{L}^{-1}(\frac{2}{s}) + \mathscr{L}^{-1}(\frac{4}{1+s}) \\
y(t) &= 2t - 2 + 4e^{-t}, \quad \text{for } t > 0
\end{aligned}$$

■ **Example 3.19** Solve the equation

$$y'' + y = \int_0^t \sin(\tau)y(t-\tau)d\tau \quad y(0) = 1,\ y'(0) = 0$$

■

Exercise 3.6 Solve

1. $y'' + y = \sqrt{2}\sin\sqrt{2}t,\ \ y(0) = 10, y'(0) = 0$
2. $y' + y = e^{-3t}\cos 2t,\ \ y(0) = 0$
3. $y' + 2y = f(t),\ y(0) = 0,\ \ f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$
4. $y' + y = \int_0^t e^{-2\tau}y(t-\tau)d\tau,\ \ y(0) = 3$
5. $y'' - y = \int_0^t \sinh(\tau)y(t-\tau)d\tau \quad y(0) = 1,\ y'(0) = 0$
6. $y'' - 4y = 2\int_0^t \sinh(2\tau)y(t-\tau)d\tau \quad y(0) = 1,\ y'(0) = 0$

■

3.8 Laplace Transform of the Integral of a function

Theorem 3.8.1 — Integration of $f(t)$ Let $F(s)$ be the Laplace transform of $f(t)$. If $f(t)$ is piecewise continuous and satisfies an inequality of the form $|f(t)| \le Me^{kt}$, then

$$\mathscr{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s), \quad (s > 0,\ s > k) \tag{3.12}$$

Or if we take the inverse transform on both sides

$$\int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\left(\frac{1}{s}F(s)\right) \tag{3.13}$$

■ **Example 3.20** Let $\mathscr{L}(f(t)) = \dfrac{1}{s^2(s^2+\omega^2)}$. Find $f(t)$ ■

Solution: We have $\mathscr{L}^{-1}\left(\dfrac{1}{s^2+\omega^2}\right) = \dfrac{1}{\omega}\sin\omega t$. From (3.13) it follows that

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) &= \frac{1}{\omega}\int_0^t \sin\omega\tau d\tau \\
&= \frac{1}{\omega}\left(-\frac{\cos\omega\tau}{\omega}\Big|_0^t\right) = \frac{1}{\omega}\left(\frac{-\cos\omega t+1}{\omega}\right) \\
&= \frac{1}{\omega^2}(1-\cos\omega t)
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{s}\left(\frac{1}{s(s^2+\omega^2)}\right)\right) &= \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\, d\tau \\
&= \frac{1}{\omega^2}\left(\tau - \frac{\sin\omega\tau}{\omega}\Big|_0^t\right) \\
&= \frac{1}{\omega^2}\left(t - \frac{\sin\omega t}{\omega}\right)
\end{aligned}$$

Electric Circuit

Consider the RLC Circuit below

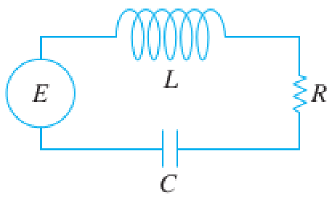


Figure 3.1: LRC series circuit.

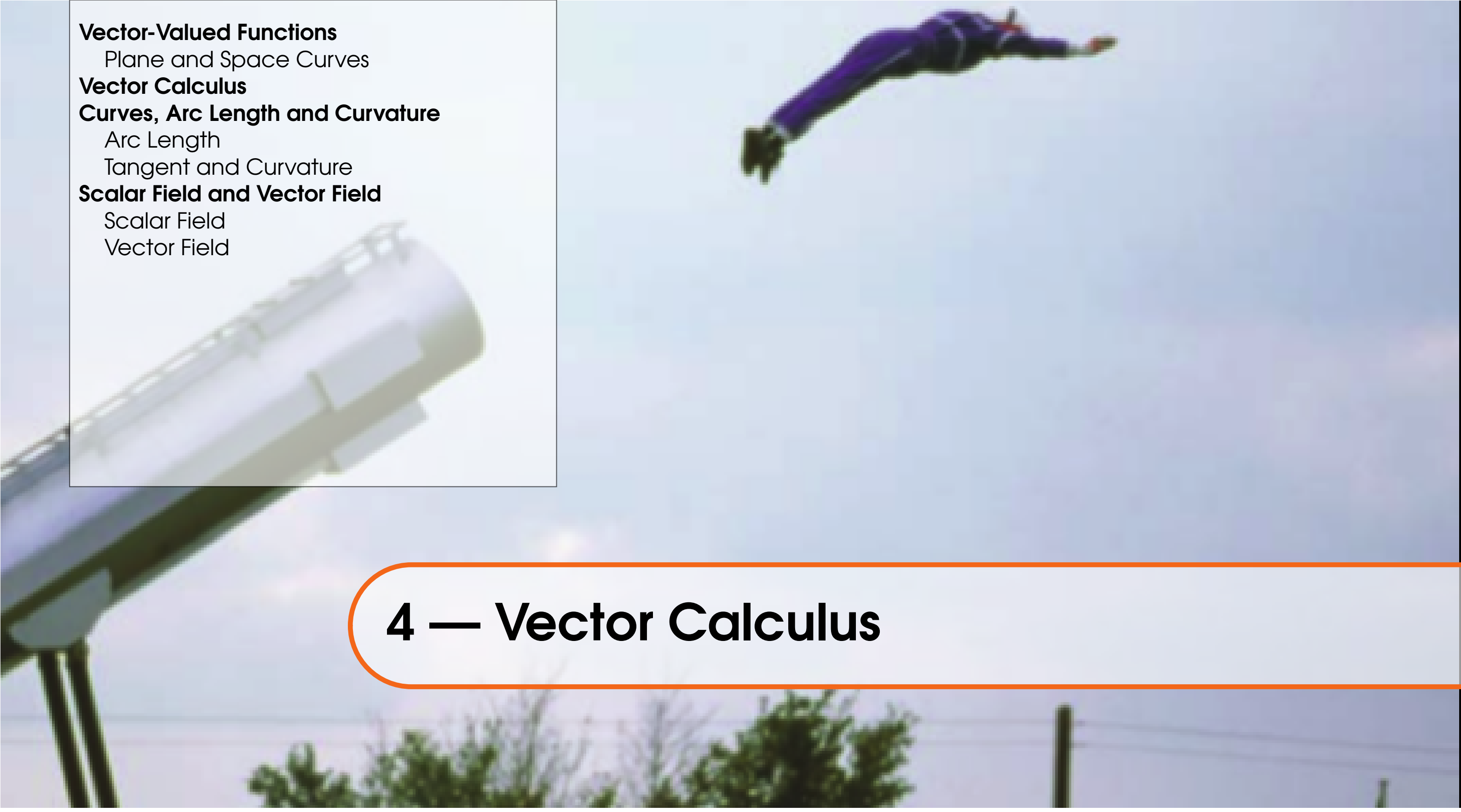
In a single-loop or series circuit, Kirchhoff's second law states that the sum of the voltage drops across an inductor, resistor, and capacitor is equal to the impressed voltage E(t). Now it is known that the voltage drops across an inductor, resistor, and capacitor are, respectively,

$$L\frac{di(t)}{dt} \quad ,Ri(\tau), \quad \text{and} \quad \frac{1}{c}\int_0^t i(\tau)d\tau$$

where $I(t)$ is the current and L, R, and C are constants. It follows that the current in a circuit, such as that shown in Figure 3.1, is governed by the **integrodifferential equation**

$$L\frac{di(t)}{dt} + Ri(\tau) + \frac{1}{c}\int_0^t i(\tau)d\tau = E(t)$$

■ **Example 3.21** Determine the current $i(t)$ in a single-loop LRC circuit when $L = 0.1\ h, R = 2\ \omega, C = 0.1\ f, i(0) = 0$, and the impressed voltage is $E(t) = 120t - 120tU(t-1)$ ■



Vector-Valued Functions
Plane and Space Curves
Vector Calculus
Curves, Arc Length and Curvature
Arc Length
Tangent and Curvature
Scalar Field and Vector Field
Scalar Field
Vector Field

4 — Vector Calculus

4.1 Vector-Valued Functions

A vector-valued function, or vector function, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

Definition 4.1.1 A vector-valued function, or vector function, is a function **r** defined by

$$r(t) = f(t)i + g(t)j + h(t)k$$

where the component functions f, g , and h of **r** are real-valued functions of the parameter t lying in a parameter interval I.

R

- Vector valued function can be used to study curves in plane or space.
- Vector valued function can be used to study the motion of an object along the curve.
- Vector valued function maps real number to vectors

Example 4.1 Find the domain (parameter interval) of the vector function

$$r(t) = \frac{1}{t}i + \sqrt{t-1}\,j + \ln t k$$

Solution: The component functions of **r** are $f(t) = \frac{1}{t}$, $g(t) = \sqrt{t-1}$ and $h(t) = \ln t$. Observe that f is defined for all values of t except $t = 0$, g is defined for all $t \geq 1$, and h is defined for all $t > 1$. Therefore, f , g , and h are all defined if $t > 1$, thus, the domain of **r** is $[1, \infty)$

4.1.1 Plane and Space Curves

Definition 4.1.2
1. A plane curve is defined as the set of ordered pairs $(f(t),\ g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$
2. A space curve is the set of all ordered triples $(f(t),\ g(t),\ h(t))$ together with their defining parametric equations $x = f(t),\ y = g(t)$ and $z = h(t)$

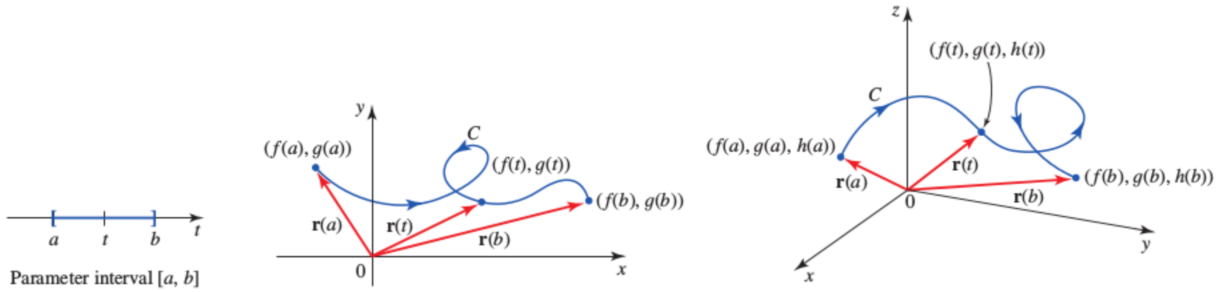


Figure 4.1: As t increases from a to b, the terminal point of r traces the curve C .

R The terminal point of position vector $r(t)$ coincides with the point $(x,\ y)$ or (x,y,z) on the curve given by the parametric equation.

The arrow head on the curve indicates the curve's orientation by pointing in the direction of increasing value of t.

■ **Example 4.2** Sketch the curve defined by the vector function

$$r(t) = 3\cos t i - 2\sin t j$$

■

Solution: The parametric equations for the curve are

$$x = 3\cos t \text{ and } y = -2\sin t$$

$$\cos t = \frac{x}{3},\ \sin t = -\frac{y}{2}$$
$$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

The curve described by this equation is the ellipse shown in Figure (4.2). As t increases from 0 to 2π , the terminal point of r traces the ellipse in a clockwise direction.

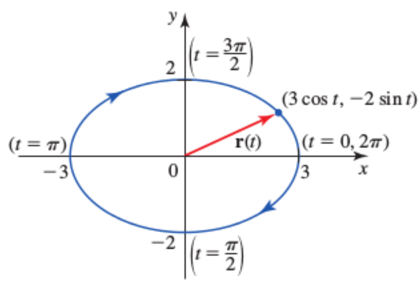


Figure 4.2:

■ **Example 4.3** Sketch the curve defined by the vector function

$$r(t) = (2-4t)i + (-1+3t)j + (3+2t)k$$

■

Solution: The parametric equations for the curve are

$$x = 2-4t,\ y = -1+3t,\ \text{and } z = 3+2t$$

which are parametric equations of the line passing through the point $(2,\ -1,\ 3)$ with direction numbers -4 , 3, and 2.

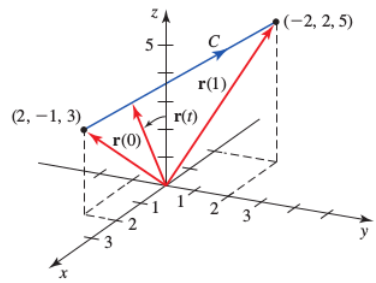
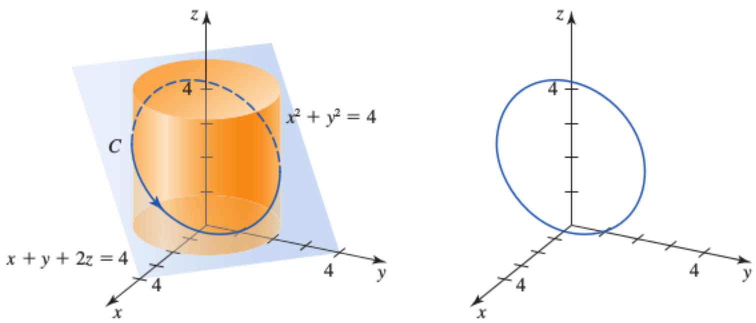


Figure 4.3:

■ **Example 4.4** Find a vector function that describes the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + 2z = 4$. ■



(a) Intersection of the plane and the cylinder **(b)** Curve of intersection

Solution : If $P(x, y, z)$ is any point on the curve of intersection C , then the x - and y -coordinates lie on the right circular cylinder of radius 2 and axis lying along the z -axis. Therefore,

$$x = 2\cos t \text{ and } y = 2\sin t$$

To find the z -coordinate of the point, we substitute these values of x and y into the equation of the plane, obtaining

$$2\cos t + 2\sin t + 2z = 4 \quad \Rightarrow \quad z = 2 - \cos t - \sin t$$

So a required vector function is

$$r(t) = 2\cos t i + 2\sin t j + (2 - \cos t - \sin t)k,\ \ 0 \le t \le 2\pi$$

4.2 Vector Calculus

Definition 4.2.1 — The Limit of a Vector Function. Let **r** be a function defined by $r(t) = f(t)i + g(t)j + h(t)k$. Then

$$\lim_{t\to a} r(t) = \left[\lim_{t\to a} f(t)\right] i + \left[\lim_{t\to a} g(t)\right] j + \left[\lim_{t\to a} h(t)\right] k$$

provided that the limits of the component functions exist.

■ **Example 4.5** Find $\lim_{t\to a} r(t)$, where $r(t) = (1+t^3)i + te^{-t}j + \frac{\sin t}{t}k$ ■

Definition 4.2.2 — Continuity of a Vector Function. A vector function **r** is continuous at a if

$$\lim_{t\to a} r(t) = r(a)$$

A vector function **r** is continuous on an interval I if it is continuous at every number in I .

Definition 4.2.3 — Derivative of a Vector Function. The derivative of a vector function **r** is the vector function **r**$'$ defined by

$$r'(t) = \frac{dr}{dt} = \lim_{h\to 0} \frac{r(t+h) - r(t)}{h}$$

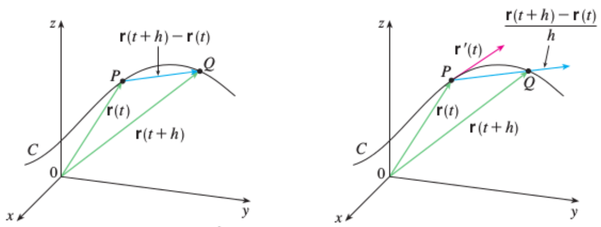
provided that the limit exists.

R

- The vector $r'(t)$ is called the **tangent vector** to the curve defined by r at the point P , provided that $r'(t)$ exists and $r'(t) \neq 0$.
- The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $r'(t)$.
- The **unit tangent vector**

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

which has unit length and the direction of r'



(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector **r**$'(t)$

Theorem 4.2.1 — Differentiation of Vector Functions Let $r(t) = f(t)i + g(t)j + h(t)k$ where f , g , and h are differentiable functions of t . Then

$$r'(t) = f'(t)i + g'(t)j + h'(t)k$$

Example 4.6 (a) Find the derivative of $r(t) = (t^2+1)i + e^{-t}j - \sin 2tk$

(b) Find the point of tangency and the unit tangent vector at the point on the curve corresponding to $t = 0$.

Solution: (a). Using the theorem, $r'(t) = 2t\ i - e^{-t}\ j - 2\cos 2t\ k$
(b). Since $r(0) = i + j$, we see that the point of tangency is $(1,1,0)$. Next, since $r'(0) = -j - 2k$, we find the unit tangent vector at $(1,1,0)$ to be

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = -\frac{1}{\sqrt{5}}j - \frac{2}{\sqrt{5}}k$$

Example 4.7 Find parametric equations for the tangent line to the helix with para- metric equations

$$x = 3\cos t,\quad y = 2\sin t,\quad z = t$$

at the point $(0,2,\pi/2)$

Solution: The vector equation of the helix is $r(t) = 3\cos t\ i + 2\sin t\ j + t\ k$, so

$$r(t) = -3\sin t\ i + 2\cos t\ j + 1\ k$$

The parameter value corresponding to the point $(0,2,\pi/2)$ is $t = \pi/2$ so the tangent vector there is $r'(\pi/2) = (-3,0,1)$. The tangent line is the line throug $(0,2,\pi/2)$ parallel to the vector $(-3,0,1)$. So its parametric equations are

$$x = -3t\quad y = 2\quad z = \frac{\pi}{2} + t$$

Theorem 4.2.2 — Rules of Differentiation Suppose that u and v are differentiable vector functions, f is a differentiable real-valued function, and c is a scalar. Then

1. $\frac{d}{dt}[u(t) \pm v(t)] = u'(t) \pm v'(t)$
2. $\frac{d}{dt}[cu(t)] = cu'(t)$
3. $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$
4. $\frac{d}{dt}[u(t)\cdot v(t)] = u'(t)\cdot v(t) + u(t)\cdot v'(t)$
5. $\frac{d}{dt}[u(t)\times v(t)] = u'(t)\times v(t) + u(t)\times v'(t)$
6. $\frac{d}{dt}[u(f(t))] = u'(f(t))f'(t)$ Chain rule

Definition 4.2.4 — Integration of Vector Functions. Let $r(t) = f(t)i + g(t)j + h(t)k$ where f , g , and h are integrable functions of t. Then

1. The indefinite integral of r with respect to t is

$$\int r(t)dt = \left[\int f(t)dt\right]i + \left[\int g(t)dt\right]j + \left[\int h(t)dt\right]k$$

2. The definite integral of r over the interval [a, b] is
$$\int_a^b r(t)dt = \left[\int_a^b f(t)dt\right] i + \left[\int_a^b g(t)dt\right] j + \left[\int_a^b h(t)dt\right] k$$

Example 4.8 Find $\int r(t)dt$ if $r(t) = (t+1)i + \cos 2t\, j + e^{3t}k$

R In general, the indefinite integral of r can be written as
$$\int r(t)dt = R(t) + C$$
where C is an arbitrary constant vector and $R'(t) = r(t)$.

4.3 Curves, Arc Length and Curvature

Definition 4.3.1 A curve C is closed if it has a parametrization whose domain is a closed interval [a, b] such that $r(a) = r(b)$.

Definition 4.3.2
1. A vector valued function **r** defined on an interval I is **smooth** if **r** has a continuous derivative on I and $r'(t) \neq 0$ for each interior point t.
   A curve C is smooth if it has a smooth parametrization.
2. A continuous vector valued function **r** defined on an interval I is **piecewise smooth** if I is composed of a finite number of subintervals on each of which r is smooth and if r has one-sided derivatives at each interior point of I.
   A curve C is piecewise smooth if it has a piecewise smooth parametrization.

Example 4.9 Show that the standard unit circle is smooth.

Solution: The circle can be parametrized by
$$r(t) = \cos t\ i + \sin t\ j, \quad \text{for} \ \ 0 \leq t \leq 2\pi$$
The function **r** is differentiable on $[0,\ 2\pi]$, and
$$r'(t) = -\sin t\ i + \cos t\ j \quad \text{for} \ \ 0 \leq t \leq 2\pi$$
Thus, r' is continuous on $[0, 2\pi]$, and
$$\|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \quad \Rightarrow \quad r'(t) \neq 0 \quad \text{for each} \ \ t \in [0, 2\pi]$$
It follows that the circle is smooth.

Example 4.10 Show that the helix
$$r(t) = \cos t\ i + \sin t\ j + t\ k$$
is smooth

Solution: Since $r'(t) = -\sin t\ i + \cos t\ j + k$, it follows that r' is continuous, and $r'(t) \neq 0$ for every t. Therefore, the helix is smooth.

■ **Example 4.11** Show tha the curve

$$r(t) = a(t - \sin t)i + a(1 - \cos t)j, \quad \text{for } -2\pi \le t \le 2\pi$$

which parametrizes two arches of a cycloid, piecewise smooth. ■

Solution: We find that

$$r'(t) = a(1t - \cos t)i + a\sin t\ j$$

$\Rightarrow$ r' is continuous but **r** is not smooth because $r'(0) = 0$.
How ever, $r'(t) \neq 0$ if t is not $-2\pi, 0, 2\pi$.
Therefore, r is smooth on $(-2\pi,\ 0)$ and on $(0,\ 2\pi)$, and hence r is piecewise smooth.

Parametrization of line segment

Suppose (x_0, y_0, z_0) and (x_1, y_1, z_1) are distinct points in spae, and consider the parametric equations

$$x = x_0 + (x_1 - x_0)t,\ y = y_0 + (y_1 - y_0)t,\ z = z_0 + (z_1 - z_0)t \tag{4.1}$$

Since $(x, y, z) = (x_0, y_0, z_0)$ at $t = 0$ and $(x, y, z) = (x_1, y_1, z_1)$ at $t = 1$, it follows that (4.1) gives parametric equation for the line through (x_0, y_0, z_0) and (x_1, y_1, z_1).
More over,

$$r(t) = [x_0 + (x_1 - x_0)t]i + [y_0 + (y_1 - y_0)t]j + [z_0 + (z_1 - z_0)t]k, \quad \text{for } 0 \le t \le 1 \tag{4.2}$$

is a smooth parametrization of the line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1).

■ **Example 4.12** Find a parametric representation of a line segment from $(3, -1, 5)$ to $(5, -5, 5)$ ■

Solution: $r(t) = (3 + 2t)i - (1 + 4t)j + 5k, \quad 0 \le t \le 1$

4.3.1 Arc Length

Definition 4.3.3 Let C be a curve with a piecewise smooth parametrization **r** defined on [a, b]. Then the length l of C is defined by

$$l = \int_a^b \|r'(t)\| dt \tag{4.3}$$

If $r(t) = x(t)\ i + y(t)\ j + z(t)\ k, a \le t \le b$ then (4.3) can be rewritten as

$$l = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

■ **Example 4.13** Find the length l of the segment of the circular helix

$$r(t) = \cos t\ i + \sin t\ j + t\ k, \quad 0 \le t \le 2\pi$$

■

Solution: $l = 2\sqrt{2}\pi$

■ **Example 4.14** Find the length l of the curve

$$r(t) = t\ i + \frac{\sqrt{6}}{2} t^2\ j + t^3\ k, \quad -1 \le t \le 1$$

■

Solution: $l = 4$

R The length l of the polar graph $r = f(\theta)$ on $[\alpha,\ \beta]$ is given by

$$l = \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta$$

■ **Example 4.15** Find the length l of the cardioid

$$r = 1 - \cos\theta \quad \text{for } 0 \le \theta \le 2\pi$$

Ans. $l = 8$ ■

R The curve C described by the vector function r(t) with parameter t in some parameter interval I is said to be **parametrized** by t. A curve C can have more than one parametrization. For example, the curve

$$r_1(t) = < t, t^2, t^3 >, \quad 1 \le t \le 2$$

could also be represented by the function

$$r_2(u) = < e^u, e^{2u}, e^{3u} > 0 \le t \le \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$.

Definition 4.3.4 Let C be a smooth curve parametrized on an interval $[a,\ b]$ by

$$r(t) = x(t)\ i + y(t)\ j + z(t)\ k, \text{ for } a \le t \le b$$

then the arc length function s is defined by

$$s(t) = \int_a^t \|r'(u)\| du = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} du \text{ for } t \in I$$

$$s'(t) = \|r'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

R $s(t)$ is the distance along C from initial point to the point $(x(t), y(t), z(t))$

■ **Example 4.16** Find the arc length function $s(t)$ for the circle C in the plane described by

$$r(t) = 2\cos t i + 2\sin t j \quad 0 \le t \le 2\pi$$

■

Solution: $s(t) = 2t \quad 0 \le t \le 2\pi \quad \Rightarrow \quad r(s) = 2\cos\left(\frac{s}{2}\right) i + 2\sin\left(\frac{s}{2}\right) j, \quad 0 \le s \le 4\pi$

R $r'(s)$ is a unit tangent vector.

4.3.2 Tangent and Curvature

Definition 4.3.5 Let C be a smooth curve and r a (smooth) parametrization of C defined on an interval I. Then for any interior point t of I, the **unit tangent vector** $T(t)$ at the point $r(t)$ is defined by

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

Example 4.17 Find the tangent vector T(t) to the circular helix

$$r(t) = 2\cos t i + 2\sin t j + 3t\ k \quad 0 \leq t \leq 2\pi$$

■

Definition 4.3.6 Let C be a smooth curve and r be a (smooth) parametrization of C defined on an interval I such that r' is smooth. Then for any interior point t of I for which $T'(t) \neq 0$, the **normal vector** $N(t)$ at the point $r(t)$ is defined by

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Example 4.18 Find the normal vector N(t) to the circle

$$r(t) = a\cos t\ i + a\sin t\ j$$

■

Curvature

- It is a measure of how sharply a curve bends.
- It is the magnitude of the rate of change of the unit tangent vector T with respect to the arc length parametr s.

Definition 4.3.7 Let C be a smooth curve parametrization **r** such that r' is differentiable. Then the curvature

$$\mathscr{K}(t) = \left\|\frac{dT}{ds}\right\| = \left\|\frac{dT}{dt}\frac{dt}{ds}\right\| = \frac{\|T'(t)\|}{\|r'(t)\|}$$

Example 4.19 Find the curvature $\mathscr{K}$ of the parabola parametrized by

$$r(t) = t\ i + t^2\ j$$

■

Solution: $\mathscr{K}(t) = \dfrac{\|T'(t)\|}{\|r'(t)\|} = \dfrac{2}{(1+4t^2)^{3/2}}$

Exercise 4.1 Show that the curvature of a circle of radius a equals $\frac{1}{a}$ ■

4.4 Scalar Field and Vector Field

4.4.1 Scalar Field

A scalar function is a function which is defined at each point of a certain set of points in space and whose values are real numbers depending only on the points in space but not on the particular choice of the coordinate system. The domain D of a scalar function may be a curve, a surface, or a three dimensional region in space.
The function f associated with each point in D a scalar, a real number, is said to be a scalar field in D.

4.4.2 Vector Field

Definition 4.4.1 Let D be a region in the plane ($\mathbb{R}^2$). A vector field in $\mathbb{R}^2$ is a vector-valued function F that associates with each point (x,y) in D a two-dimensional vector

$$F(x,y) = P(x,y)i + Q(x,y)j$$

where P and Q are functions of two variables defined on $\mathbb{R}^2$.